10/14/2016 | Intro to Young Tableau
Fulton, Ch. 8

$V = \mathbb{C}^m$

$V \otimes \cdots \otimes V$

Ex: $V \otimes V = Sym^2(V) \oplus \Lambda^2 V$

$Sym^2(V) = Span\{v \otimes w + w \otimes v\}$

Basis: $e_i \otimes e_j + e_j \otimes e_i$, $e_i \otimes e_i$

$\dim = \frac{m(m-1)}{2} + m = \frac{m(m+1)}{2}$

$\Lambda^2 V = Span\{v \otimes w - w \otimes v\}$

Basis: $e_i \otimes e_j - e_j \otimes e_i$

$\dim = \frac{m(m-1)}{2}$

$m^2 = \frac{m(m+1)}{2} + \frac{m(m-1)}{2}$

Note: ① $\langle \cdot, \cdot \rangle$ usual HIP on $\mathbb{C}^m$

induced HIP on $V \otimes \cdots \otimes V$

$\langle u_1 \otimes \cdots \otimes u_k, v_1 \otimes \cdots \otimes v_k \rangle$
$= \langle u_1, v_1 \rangle \cdots \langle u_k, v_k \rangle$

$\rightarrow Sym^2(V) \perp \Lambda^2 V$

② $GL(m, \mathbb{C})$ action $V \circlearrowleft V$ as usual

$g \cdot v = gv$ (matrix vector product)

$g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$

Note: $Sym^2(V) \subseteq V \otimes V$ invariant under $GL(m, \mathbb{C})$ action (linear)

$g(v \otimes w + w \otimes v) = gv \otimes gw + gw \otimes gv \in Sym^2(V)$

Likewise for $\Lambda^2 V \subseteq V \otimes V$.

③ $S_2 = \mathbb{Z}/2$ - linear action on $V \otimes V$

$\sigma(v \otimes w) = w \otimes v$ (or $-w \otimes v$)

by 1 on $Sym^2(V)$ (by -1)

by -1 on $\Lambda^2 V$ (by $+1$)

Try $V = \mathbb{C}^3$ $V \otimes V \otimes V$ $\dim = 27$.

$\Lambda^3 V = span\{e_1 \wedge e_2 \wedge e_3 = \sum_{\sigma} (-1)^{\sigma} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}\}$

$\dim \Lambda^3 V = 1$

$Sym^3(V) = Span\{e_i^3, e_i^2 e_j, e_i e_j e_k\}$
$3 + 6 + 1 = 10$

$e_i^3 = e_i \otimes e_i \otimes e_i$

$e_i^2 e_j = e_i \otimes e_i \otimes e_j + e_i \otimes e_j \otimes e_i + e_j \otimes e_i \otimes e_i$

$e_1 \cdot e_2 \cdot e_3 = \sum_{\sigma} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$

Missing 16 dimensions?

$\boxed{112 - 121} \subseteq V \otimes \Lambda^2 V$ | $\boxed{211 - 121} \subseteq \Lambda^2 V \otimes V$

$= e_1 \otimes (e_1 \wedge e_2)$

$\boxed{312 + 132 - 321 - 123}$ | $\boxed{213 + 231 - 123 - 321}$

$= e_3 \otimes (e_1 \wedge e_2) + e_1 \otimes (e_3 \wedge e_2)$

113 − 131	311 − 131
213 − 231	312 − 132
332 − 323	233 − 323
313 − 331	313 − 133
212 − 221	212 − 122
232 − 223	232 − 322

How to find these:

① Ortho to $\Lambda^3 V$, $Sym^3 V$

② Invariant under $GL(m, \mathbb{C})$

③ $\mathbb{Z}/2$ action: Monomials + signs.

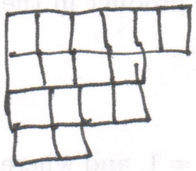
Tableau: $n \geq 1$ boxes

- Diagram : boxes
- Tableau : numbered w/repeats, inc → col, weak inc → row
- Standard Tableau : no repeats

$\lambda = (6,4,4,2)$ (weak dec)

Partition: $16 = 6+4+4+2$

6
4
4
2



Tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&3&5\\\hline 2&3&5&5\\\cline{1-4} 4&4&6&6\\\cline{1-4} 5&6\\\cline{1-2}\end{array}$$

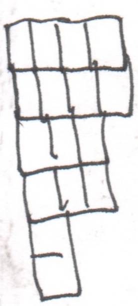
Standard Tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\\hline 7&8&9&10\\\cline{1-4} 11&12&13&14\\\cline{1-4} 15&16\\\cline{1-2}\end{array}$$

Conjugate: $\tilde{\lambda}$

(transpose)

- diagram ✓
- tableau ✗
- std tableau ✓



$\lambda \rightarrow \tilde{\lambda}$

$(6,4,4,2)$ $(4,4,3,3,1,1)$

Schur Polynomials: (always homog, symmetric)

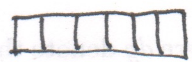
λ, m

λ has $\leq m$ parts (rows)

$$S_\lambda(x_1 \dots x_m) = \sum x^T$$

$T \rightarrow x^T$ monomial

$$\text{tableau} = \prod_i (x_i)^{\text{\# times } i \text{ is in } T}$$

$\lambda = (n)$  (1 part)

n^{th} complete symmetric polynomial in $x_1 \dots x_m$

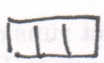
$n=2$: 

$\lambda = (2)$ $m=2$: $x_1\, x_2$ $\boxed{1|1}$ $\boxed{1|2}$ $\boxed{2|2}$

$$x_1^2 + x_1x_2 + x_2^2$$

$m=3$: $x_1\,x_2\,x_3$ $\boxed{1|1}$ $\boxed{2|2}$ $\boxed{3|3}$ $\boxed{1|2}$ $\boxed{2|3}$ $\boxed{1|3}$

$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3$$

$\lambda = (3)$:  $\boxed{1|1|1}$ ... $\boxed{2|2|2}$

$m=4$: $x_1\,x_2\,x_3\,x_4$ $\boxed{1|1|2}$ ---

$\boxed{1|2|3}$ ---

Ex: $\lambda = (1, \ldots, 1) = (1^n)$ n^{th} elementary symmetric polynomial

$n=3$

$m=3$ $x_1 x_2 x_3$ $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

$x_1 x_2 x_3$

$n=3$ $x_1 x_2 x_3 x_4$

$m=4$

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array}$ $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$ $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$

$x_1 x_2 x_3 + x_1 x_2 x_4 + x_2 x_3 x_4 + x_1 x_3 x_4$

Ex: $\lambda = (2,1)$ $\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

$m=3$

$x_1 x_2 x_3$

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

$x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_2 x_3^2 + x_1 x_3^2 + x_1 x_2 x_3$

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Last time: tensors

$Sym^2(V)$, $\Lambda^2 V$, $V \otimes V$, $V = \mathbb{C}^m$

universal mapping property:

① $V \otimes W$ All symbols of form: $\sum_i v_i \otimes w_i$

① $v_1 + v_2 \otimes w = v_1 \otimes w + v_2 \otimes w$

② $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$

UMP: $f: V \oplus W \to U$ multilinear

factors $V \oplus W \underset{\pi \text{ linear}}{\to} V \otimes W \underset{\tilde{f} \text{ linear}}{\to} U$

② $Sym^2(V)$

UMP: $f: V \oplus V \to U$

① multilinear

② $f(v_1 v_2) = f(v_2 v_1)$

Then $V \oplus V \underset{\pi}{\to} Sym^2(V) \underset{\tilde{f}}{\to} U$

Multilinear:

$V \oplus V \underset{\tilde{\pi}_1}{\to} V \otimes V \underset{\pi_2}{\to} Sym^2(V) \underset{\tilde{f}}{\to} U$

$Sym^2(V) \simeq V \otimes V / I_1 (= \langle v \otimes w - w \otimes v \rangle)$

$= \Lambda^2 V$

③ $\Lambda^2 V$

$V \oplus V \to W$

UMP: ① multilinear

② $f(v,w) = -f(w,v)$

$V \oplus V \to \Lambda^2 V \to W$

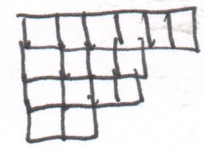
$V \oplus V \to V \otimes V \to \Lambda^2 V \to W$

$\Lambda^2 V \simeq V \otimes V / I_2 (= \langle v \otimes w + w \otimes v \rangle)$

$= Sym^2(V)$

General Construction:

Tableau: partition of n, $\le m$ rows

$\lambda = (6,4,4,2)$: 

numbering any $1 \to m$ — inc. = along rows, inc along columns

1	2	2	3	3	5
2	3	5	5		
4	4	6	6		
5	6				

E^λ Schur module $|\lambda| = n$

universal mapping property wrt

tableau records ~~partition~~ positions in $\underbrace{V \otimes \cdots \otimes V}_{n}$

Rules: ① multilinear

② Interchange boxes (pair) in column $\to \times -1$

③ Row interchange:

pick ~~2~~ columns

top k in right column.

All k interchanges into left column (see ex.)

$f(T) = \sum f(T')$

$f: \overbrace{V \oplus \cdots \oplus V}^{n} \to W$ w/ ①-③

$f: V \oplus \cdots \oplus V \to \Lambda^{c_1} V \otimes \cdots \Lambda^{c_a} V$

$\to E^\lambda \to W$

Ex: $\lambda = \begin{array}{|c|}\hline \ \\\hline \ \\\hline \ \\\hline \ \\\hline \ \\\hline\end{array} = (1,\dots,1) = (1^n)$ $\quad n \le m$

$f: \bigoplus^n V \to W$ $\qquad \Lambda^n V$

① multi

② all row switches

③ does not apply

$\begin{array}{|c|}\hline u_1\\\hline u_2\\\hline \vdots\\\hline u_n\\\hline\end{array}$: $f(u_1\, u_i u_j\, u_n) = -f(u_1 u_j u_i u_n)$ $\quad \begin{array}{|c|}\hline \ \\\hline u_j\\\hline u_i\\\hline \ \\\hline\end{array}$

$\Rightarrow \tilde f(u_1 \otimes u_i u_j \otimes u_n + u_1 \otimes u_j u_i \otimes u_n) = 0$

$I = \langle u_1 \otimes u_i u_j \otimes u_n + u_1 \otimes u_j u_i \otimes u_n \rangle$

all i,j

Ex: $\lambda = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\\hline\end{array} = (n)$

$f: \bigoplus^n V \to W$

① multilinear

② no col switches

③ → switch any 2 boxes

$f\left(\begin{array}{|c|c|c|c|}\hline \ & i & j & \ \\\hline\end{array}\right) = f\left(\begin{array}{|c|c|c|c|}\hline \ & j & i & \ \\\hline\end{array}\right)$

$f(u_1 \dots u_i\, u_j \dots u_n) = f(u_1\, u_j\, u_i\, u_n)$

$\tilde f(u_1 \otimes u_i\, u_j \otimes u_n - u_1 \otimes u_j u_i \otimes u_n) = 0$

$I = \langle u_1 \otimes u_i u_j \otimes u_n - u_1 \otimes u_j u_i \otimes u_n \rangle$

all i,j

Ex: $\lambda = (2,2)$ $\qquad$ label inc. down cols. first

② $f\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\right) = -f\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array}\right) = -f\left(\begin{array}{|c|c|}\hline 1&4\\\hline 2&3\\\hline\end{array}\right)$

③ $f\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\right) = f\left(\begin{array}{|c|c|}\hline 3&1\\\hline 2&4\\\hline\end{array}\right) + f\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right)$

$f\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\right) = f\left(\begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}\right)$

Ex: $\lambda = (2,2,2)$

$f\left(\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}\right) = f\left(\begin{array}{|c|c|}\hline 4&1\\\hline 2&5\\\hline 3&6\\\hline\end{array}\right) + f\left(\begin{array}{|c|c|}\hline 1&2\\\hline 4&5\\\hline 3&6\\\hline\end{array}\right) + f\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array}\right)$

$f\left(\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}\right) = f\begin{array}{|c|c|}\hline 4&1\\\hline 5&2\\\hline 3&6\\\hline\end{array} + f\begin{array}{|c|c|}\hline 4&1\\\hline 2&3\\\hline 5&6\\\hline\end{array} + f\begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline 5&6\\\hline\end{array}$

$f\left(\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}\right) = f\left(\begin{array}{|c|c|}\hline 4&1\\\hline 5&2\\\hline 6&3\\\hline\end{array}\right)$

Ex: $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ $\qquad \lambda = (2,1)$

① multi

② $f(u_1 u_2 u_3) = -f(u_2 u_1 u_3)$ $\quad \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$

③ $f(u_1 u_2 u_3) = f(u_3 u_2 u_1)$ $\quad \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}$

$\qquad + f(u_1 u_3 u_2)$ $\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$

$I = \langle u_1 \otimes u_2 \otimes u_3 + u_2 \otimes u_1 \otimes u_3,$
$\quad u_1 \otimes u_2 \otimes u_3 - u_3 \otimes u_2 \otimes u_1$
$\quad - u_1 \otimes u_3 \otimes u_2 \rangle$

Note: I cont. $123 + 213$,

$213 + 321 + 132$

$V \otimes V \otimes V \to \Lambda^2 V \otimes V \to E^\lambda$
$\qquad\qquad u_1 \wedge u_2 \quad u_3$

10/14/2016 $\begin{array}{|c|c|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array}$ → piece generated by $211 - 121$

8 dimensional

10/25/2016

Revisit $V\otimes V\otimes V$ w/ $V=\mathbb{C}^3$

$\Lambda^2 V\otimes V \subseteq V\otimes V\otimes V$

$\dim = \underline{3\cdot 3 = 9}$

$\mathrm{Span}\{(u\wedge v)\otimes w\} = W_1$

$V\otimes \Lambda^2 V \subseteq V\otimes V\otimes V$

$\underline{3\cdot 3 = 9}$

$\mathrm{Span}\{u\otimes(v\wedge w)\} = W_2$

$W_1 \cap W_2 = \Lambda^3 V$

$$\underset{9}{\Lambda^2 V\otimes V} = \underset{8}{\ydiagram{2,1}} \oplus \underset{1}{\Lambda^3 V}$$

$$\underset{9}{V\otimes \Lambda^2 V} = \underset{\text{"8"}}{\ydiagram{2,1}} \oplus \underset{1}{\Lambda^3 V}$$

$$\boxed{V\otimes V\otimes V = \Lambda^3 V \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \mathrm{Sym}^3(V)}$$

LI

but not ortho

Recipe for finding $\ydiagram{2,1}$ basis:

Notation $e_i\otimes e_j\otimes e_k = ijk$ (order matters)

	lower	raise
6 ops	$1\to 2$	$2\to 1$
	$2\to 3$	$3\to 1$
	$1\to 3$	$3\to 2$

On V: $\to 0$ if no match

ex: $1\to 2: 223 \to 0$

On $V\otimes\ldots\otimes V$: Product rule.

① Find v s.t.

all $\begin{matrix}2\to 1\\ 3\to 1\\ 3\to 2\end{matrix} : v = 0$ (highest weight vector)

② Repeatedly apply $\begin{matrix}1\to 2\\ 2\to 3\\ 1\to 3\end{matrix}$ until $=0$.

Ex: $\boxed{112-121}$ highest weight vector

$2\to 1: 111-111=0$

$3\to 1: 0-0=0$

$3\to 2: 0-0=0$

$112-121$:

$1\to 2: 212+122-221-122 = \boxed{212-221}$

$1\to 3: \boxed{312+132-321-123}$

$2\to 3: \boxed{113-131}$

$212-221$: $1\to 3: \boxed{232-223}$

$2\to 3: \boxed{332-323}$

$2\to 3: 313+133-331-133$

$= \boxed{313-331}$

$1\to 2: \boxed{213+123-231-132}$

$\Rightarrow$ 8 basis vectors. (all ortho.)

Involution: $u\otimes v\otimes w \to w\otimes v\otimes u$

sends $\Lambda^2 V\otimes V \longleftrightarrow V\otimes\Lambda^2 V$

$211-121$	hwv
$212-122$	
$213+231-123-321$	
$311-131$	
$232-322$	
$233-323$	
$313-133$	
$312+321-132-231$	

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Last time: tableau $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$

$f(u_1 u_2 u_3)$

1. switch 2 boxes in col $\to x-1$
2. switch & boxes between 2 cols.
   & top of right col.
   $\to$ all poss in left col in order

Ex: $\begin{array}{|c|c|c|c|}\hline u_1 & u_3 & u_4 & \cdots \\ \hline u_2 & & & \\ \cline{1-1}\end{array}$ $\quad \lambda = (n-1, 1)$

1. f multilinear
2. $f(u_1 u_2 u_3 \cdots) = -f(u_2 u_1 \cdots)$
3. $f\left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & \\ \hline 2 & & & \\ \cline{1-1}\end{array}\right) = f\left(\begin{array}{|c|c|c|c|}\hline i & \cdots & 1 & \\ \hline 2 & & & \\ \cline{1-1}\end{array}\right) + f\left(\begin{array}{|c|c|c|c|}\hline 1 & \cdots & 2 & \\ \hline i & & & \\ \cline{1-1}\end{array}\right)$

$f(u_1 u_2 u_3 u_i) = f(u_i u_2 \cdots u_1 \cdots)$
$\qquad + f(u_1 u_i \cdots u_2 \cdots)$

$I = \langle u_1 \otimes u_2 \otimes u_3 \otimes \cdots + u_2 \otimes u_1 \otimes u_3 \otimes \cdots,$
$\quad u_1 u_2 u_3 \cdots \otimes u_i \otimes \cdots - u_i \otimes u_2 \otimes \cdots \otimes u_1 \otimes \cdots$
$\qquad - u_1 \otimes u_i \otimes \cdots \otimes u_2 \otimes \cdots \rangle$

$E^\lambda =$

UMP: $V \otimes \cdots \otimes V \to \underset{\substack{\uparrow \\ \text{each col} \\ \text{corresp to } \Lambda^{c_i} V}}{\Lambda^2 V} \otimes V \otimes \cdots \otimes V \to E^\lambda \to W$

$E^\lambda = V \otimes \cdots \otimes V / I$
$\quad = \Lambda^{c_1} V \otimes \cdots \otimes \Lambda^{c_a} V / I'$

1. Can realize E^λ inside $V \otimes \cdots \otimes V = E^\lambda \oplus I$
   $\to$ first $\Lambda^{c_1} V \otimes \cdots \otimes \Lambda^{c_a} V = V'$ $\quad$ using $\langle \cdot, \cdot \rangle$
   $\to$ ortho to $V' \cap I = I'$ $\quad$ on $V \otimes \cdots \otimes V$

Example: $V \otimes V \otimes V$

$= \Lambda^3 V \oplus \mathrm{Sym}^3(V)$
$\quad \oplus \begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array}$

Perp to $\Lambda^3 V$ in $\Lambda^2 V \otimes V$, $V \otimes \Lambda^2 V$
$\quad \dim = 1 \qquad 9 = 8 + 1 \qquad 9 = 8 + 1$

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2. $GL(m, \mathbb{C})$ action on $V \otimes \cdots \otimes V$

$g(v_1 \otimes \cdots \otimes v_n) = g v_1 \otimes \cdots \otimes g v_n$

Each I is $GL(m, \mathbb{C})$ invariant
(only position matters)

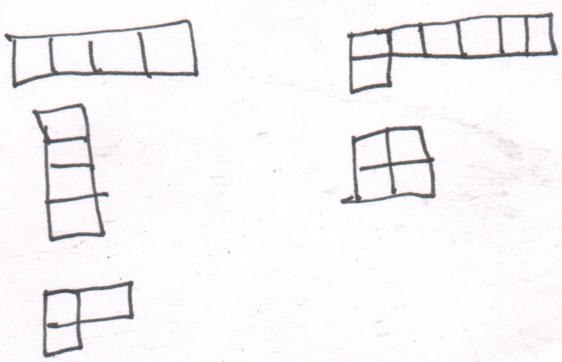
so $E^\lambda = V \otimes \cdots \otimes V / I$
admits a $GL(m, \mathbb{C})$-action
$\to$ rep'n theory of $GL(m, \mathbb{C})$

Also admits $GL(m, \mathbb{C})$ when
realized as summand in $V \otimes \cdots \otimes V$.
$= E^\lambda \oplus I$

---

3. Basis of E^λ:
   $\to$ all tableau of shape λ
   with entries $1, \ldots, m$.

   (a) col's $\to$ distinct since $\Lambda^{c_i} V$
   (b) can always push boxes to left $\to$ all larger at right



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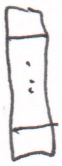
Later: Hook length formula for dimension of E^λ (special case of Weyl Dimension Formula)

Basic Examples: dim = # tableau of shape λ, filled with $1, \dots, m$

$V = \mathbb{C}^m$

1  $k (\le m)$ boxes:

choose k distinct $1, \dots, m$

order down column

$\Lambda^k \mathbb{C}^m$ $\rightarrow$ $\boxed{\dim(\Lambda^k \mathbb{C}^m) = \binom{m}{k}}$

2  k boxes

$Sym^k(\mathbb{C}^m)$

Assign each box elt of $1, \dots, m$.

Then order. So

placing k balls into m boxes.

$\rightarrow$ sequences of k 0s, $m-1$ 1s.

(remainder goes to last box)

$\boxed{\dim Sym^k(\mathbb{C}^m) = \binom{m+k-1}{m-1}}$

Note: $E^{(2,1)}$ can be realized as $\perp$ of $\Lambda^3 V$ in $\Lambda^2 V \otimes V$

1 $\dim \Lambda^3 V = \frac{m(m-1)(m-2)}{6}$

2 $\dim \Lambda^2 V \otimes V = \frac{m(m-1)}{2} \cdot m$

$$2 - 1 = \frac{m^2(m-1)}{2} - \frac{m(m-1)(m-2)}{6}$$

$$= \frac{m(m-1)}{6} \big[\underbrace{3m - (m-2)}_{2m+2}\big]$$

$$= \frac{m(m-1)(m+1)}{3}$$

so $\boxed{\Lambda^2 V \otimes V = \Lambda^3 V \oplus E^{(2,1)}}$

3  3 boxes $\boxed{\dim E^{(2,1)}}$ $\boxed{= \frac{m^3 - m}{3}}$

Suppose 

$1 \le k \le m-1$.

$m-k+1$ choices to right box ($\ge$)

$m-k$ choices to below box ($>$)

$$\sum_{k=1}^{m-1} (m-k+1)(m-k) = \sum m^2 + m - (2m+1)k + k^2$$

$$= m(m-1)(m+1) - (2m+1)\frac{(m-1)(m)}{2} + \frac{(m-1)m(2m-1)}{6}$$

$$\sum_{k=1}^{i} i = \frac{i(i+1)}{2}$$

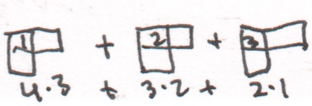
$$\sum_{k=1}^{i} i^2 = \frac{i(i+1)(2i+1)}{6}$$

$$= \frac{m(m-1)2(m+1)}{6} = \frac{m(m-1)(m+1)}{3}$$

$$= \frac{m^3 - m}{3} \checkmark$$

Check: $m=3$ $\dim E^\lambda = 8 \checkmark$

$m=4$: $\dim E^\lambda = 20 \checkmark$



$= 20$

$4 \cdot 3 + 3 \cdot 2 + 2 \cdot 1$

11/1/2016

Start of invariant Theory

Sylvester's lemma: M, N $p \times p$

(1851) $\det(M)\det(N) = \sum \det(M')\det(N')$

sum: fix k. fix first k columns of N.

M' = any switch of 1^{st} k cols of N into M in order.

N' = replace w/ switched cols of M' in first k cols in order.

Ex: 2×2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} x & y \\ z & w \end{vmatrix} = \begin{vmatrix} x & b \\ z & d \end{vmatrix}\begin{vmatrix} a & y \\ c & w \end{vmatrix} + \begin{vmatrix} a & x \\ c & z \end{vmatrix}\begin{vmatrix} b & y \\ d & w \end{vmatrix}$

$(ad-bc)(xw-yz) = (xd-bz)(aw-cy) + (az-cx)(bw-dy)$ ✓

Ex: 3×3: $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x_2 & x_3 \\ 0 & y_2 & y_3 \\ 0 & z_2 & z_3 \end{vmatrix}\begin{vmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & 1 & x_3 \\ y_1 & 0 & y_3 \\ z_1 & 0 & z_3 \end{vmatrix}\begin{vmatrix} x_2 & 0 & 0 \\ y_2 & 1 & 0 \\ z_2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 0 \\ z_1 & z_2 & 0 \end{vmatrix}\begin{vmatrix} x_3 & 0 & 0 \\ y_3 & 1 & 0 \\ z_3 & 0 & 1 \end{vmatrix}$

(row expand det along row 1 of M) $= z_3(x_1y_2 - x_2y_1) - (z_2)(x_1y_3 - x_3y_1) + z_1(x_2y_3 - x_3y_2)$

$= \det(M)$

$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & x_3 \\ 0 & 1 & y_3 \\ 0 & 0 & z_3 \end{vmatrix}\begin{vmatrix} x_1 & x_2 & 0 \\ y_1 & y_2 & 0 \\ z_1 & z_2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & x_2 & 0 \\ 0 & y_2 & 1 \\ 0 & z_2 & 0 \end{vmatrix}\begin{vmatrix} x_1 & x_3 & 0 \\ y_1 & y_3 & 0 \\ z_1 & z_3 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & 1 & 0 \\ y_1 & 0 & 1 \\ z_1 & 0 & 0 \end{vmatrix}\begin{vmatrix} x_2 & x_3 & 0 \\ y_2 & y_3 & 0 \\ z_2 & z_3 & 1 \end{vmatrix}$

$= x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1)$

$= \det(M)$

switch

Easy Cases: (1) M, N diagonal

(2) M, N both upper tri, lower tri

[here only nonzero term on LHS is first k N → first k M

otherwise, 0 on diag after permuting columns. in order by height.]

(3) After 3x3 above: M any. N diagonal.

any size now $n \times n$ (again $k=1$, row expansion of det along row 1)

(Return to SL later!)

11/1/2016

Basic Representation Theory:

For now, groups of interest (matrix multiplication)

$GL(m,\mathbb{C}) = \{m \times m \text{ matrices}/\mathbb{C},\ \det \neq 0\}$

$SL(m,\mathbb{C}) = \{\det = 1\}$

$U(m) = \{MM^* = I\}$ or $= \{M^{-1} = M^*\}$ or $= \{(M^*)^{-1} = (M^{-1})^* = M\}$

① $(MN)^* = N^*M^* = N^{-1}M^{-1} = (MN)^{-1}$ — More later!

② $I^* = I = I^{-1}$

③ Assoc. (True for matrix mult)

④ $M^{-1} = M^*$ then $(M^{-1})^* = (M^*)^{-1} = (M^{-1})^{-1}$

Definitions: G arbitrary for now.

① (π, V) representation: $\pi : G \xrightarrow[\text{hom}]{} GL(V)$ — "group action of G on V by linear transformations"

V v.s./$\mathbb{C}$, finite dim'l — $\left(\pi(gh) = \pi(g)\pi(h),\ \pi(e) = I \right)$

---

② π unitary

$\langle \cdot, \cdot \rangle$ HIP on V

all g in G, u,v in V: $\langle \pi(g)u, \pi(g)v \rangle = \langle u, v \rangle$ — "G acts by isometries"

---

③ π irreducible: only invariant subspaces are $0, V$.

(W invariant: if $w \in W$, $g \in G$ then $\pi(g)w \in W$ also)

Choose basis for W, complete to basis for V

then $[\pi(g)]_B = \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]$ block upper tri.

---

④ π fully reducible: if W invariant in V, then $\exists\, W'$ in V also invariant such that $W + W' = V$ $(W \cap W' = 0)$

(Equiv. $V = W_1 \oplus \cdots \oplus W_k$, each W_i invariant, irreducible.)

If choose basis for each $W_i \Rightarrow$ basis for V then $[\pi(g)]_B = \left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array}\right]$ block diagonal.

Example: ① $V = \mathbb{C}^m$ irreducible under $GL(m,\mathbb{C})$, $U(m)$, $SL(m,\mathbb{C})$ — $\pi(g)v = gv$ (matrix-vector mult)

② $V \otimes V = \Lambda^2 V \oplus \mathrm{Sym}^2(V)$ both invariant, irreducible under $GL(m,\mathbb{C})$

$\pi(g)(v \otimes w) = gv \otimes gw$

③ $V \otimes V \otimes V = \Lambda^3 V \oplus \quad \oplus \quad \oplus \mathrm{Sym}^3(V)$

$\pi(g)(u \otimes v \otimes w) = gu \otimes gv \otimes gw$

11/8/2016

Sylvester's lemma (1851):

$$\det(M)\det(N) = \sum \det(M')\det(N')$$

sum over all switches:
Fix $k>0$. First k cols $N \to$ any k cols of M in order.

Proof #1: let $F(\overset{M}{m_1 \dots m_p}\ \overset{N}{n_1 \dots n_p}) = \sum \det(M')\det(N')$

$$F: \mathbb{C}^p + \dots + \mathbb{C}^p \to \mathbb{C}$$

F is ① multilinear
② alternating in first p, in second p

By ①, enough to evaluate on $F(e_{i_1} \dots e_{i_p}\, e_{j_1} \dots e_{j_p})$

By ②, enough to evaluate $F(I, I)$.

Agrees with $\det(M)\det(N) = F_2(M, N)$, so $F = F_2$ //
① multilinear
② alt in 1st p, 2nd p. //

(Bad $\to$ depends on choice of basis)

Proof #2: F above

$$\underbrace{\mathbb{C}^p \oplus \dots \oplus \mathbb{C}^p}_{2p} \to \mathbb{C}$$

UMP for $\otimes$, $\Lambda^p \mathbb{C}^p$ (x2)

$$\tilde{F}: \underbrace{\underbrace{\Lambda^p \mathbb{C}^p}_{\text{1-dim}} \otimes \underbrace{\Lambda^p \mathbb{C}^p}_{\text{1-dim}}}_{\text{1-dim}} \to \mathbb{C} \quad \text{linear}$$

So $\tilde{F} = c \det(M)\det(N)$

Solve for $\underset{(=1)}{c}$ using $F(I, I) = 1$. //

Proof #3: Both sides can be thought of as polynomials in matrix entries. ...

Note: 1st yr grad algebra. (UMP of $\Lambda^k \mathbb{C}^p$)

$F(m_1 \dots m_p) =$ ① multilinear
② alternating

$\tilde{F}: \Lambda^p \mathbb{C}^p \to \mathbb{C}$ linear

$\Rightarrow \tilde{F}(m_1 \dots m_p) = c \det(M)$.

11/8/2016

Back to Rep Theory:

Definitions:

(π, V) representation

$\langle \cdot, \cdot \rangle$ HIP $\to$ unitary

invariant subspace

irreducible

Examples: $G = GL(m, \mathbb{C})$

① Trivial rep $\pi : GL(m, \mathbb{C}) \to \mathbb{C}^*$

$\pi(g)v = v$ (or $\pi(g) = 1$)

② $\det : GL(m, \mathbb{C}) \to \mathbb{C}^*$

$g \mapsto \det(g)$

$\det(gh) = \det(g)\det(h)$ ✓

Also: $k \in \mathbb{Z}$ $\det^k : GL(m, \mathbb{C}) \to \mathbb{C}^*$

$g \mapsto [\det(g)]^k$

Tableau?

Interpret space for ② as

$V = \Lambda^m \mathbb{C}^m$ ($\dim = 1$)

Then $g(e_1 \wedge \dots \wedge e_m) = ge_1 \wedge \dots \wedge ge_m$

$= \det(g)(e_1 \wedge \dots \wedge e_m)$

For $\det^k$, interpret space as

$$V = \underbrace{\Lambda^m \mathbb{C}^m \otimes \dots \otimes \Lambda^m \mathbb{C}^m}_{k\text{-factors}}$$

then $g((e_1 \wedge \dots \wedge e_m) \otimes \dots \otimes (e_1 \wedge \dots \wedge e_m))$

$= (\det(g))^k (e_1 \wedge \dots \wedge e_m) \otimes \dots \otimes (e_1 \wedge \dots \wedge e_m)$

In terms of tableau, associated space is quotient of

$\Lambda^{c_1} \mathbb{C}^m \otimes \Lambda^{c_2} \mathbb{C}^m \otimes \dots \otimes \Lambda^{c_r} \mathbb{C}^m$.

If any columns of length m, we can split off as det factor.

Ex: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}$ $\mathbb{C}^2$ $m=2$

$\to \Lambda^2 \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^2 = \det^2$

$G = GL(2, \mathbb{C})$

Often convenient to assume tableau with $< m$ rows for this reason.

---

③ Standard rep'n of $GL(m, \mathbb{C})$ on $\mathbb{C}^m$

Usual matrix-vector product

$\pi(g)v = gv$.

Tools to understand irred rep'ns of $GL(m, \mathbb{C})$:

Restrict to subgroups

① T (tori) $=$ diagonal matrices $= \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}$

$= \mathbb{C}^* \times \dots \times \mathbb{C}^*$

② B (Borel) $=$ upper/lower triangular subgroups

③ W (Weyl) $=$ permutation matrices $\simeq S_m$

④ $U(m)$ unitary group $= \{MM^* = I\}$ unitary matrices

① T = diagonals

$$t = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_m \end{bmatrix}$$

$\pi(t_1 \dots t_m)$

π finite dimensional

$\pi(t)$ are simultaneously diagonalizable

→ joint eigenvalues of form

$\pi(t) = t_1^{k_1} \cdots t_m^{k_m}$

→ $(k_1, \dots, k_m)$ <u>weight</u>

$\in \mathbb{Z}^m$ (= tuple of exponents)

→ <u>weight set</u>

<u>Example</u>: $\det(t) = t_1^1 \cdots t_m^1$

→ $\boxed{(1,1,\dots,1,1)}$ $\quad \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \vdots \\ \hline m \\ \hline \end{array}$

<u>Example</u>: $\det^k(t) = t_1^k \cdots t_m^k$

$k>0$ → $\boxed{(k, \dots, k)}$ $\quad \underbrace{\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2&2&2\\ \hline \cdot&\cdot&\cdot&\cdot \\ \hline m&m&m&m \\ \hline \end{array}}_{k}$

<u>Example</u>: $V = \mathbb{C}^m$ $\quad \boxed{i}$

$\pi(t) e_i = t_i e_i \to (0 \dots \underset{i}{1} \dots 0)$

$$\to \boxed{\begin{matrix}(1,0\dots 0)\\ (0\,1\,0\dots 0)\\ (0\,0\,1\dots 0)\\ (0\,0\,0\dots 1)\end{matrix}}$$

<u>Example</u>: $V \otimes V$

$\pi(t)(e_i \otimes e_j) = (t_i^1 t_j^1)\, e_i \otimes e_j$

→ either $(0\,1\,0\,0\,1\,0)$ (mult 2)

or $(0\,0\,2\,0\,0)$ (mult 1)

Note: all weights live on hyperplane $\sum x_i = k$ in $\mathbb{Z}^m$ ←

In general, $\overbrace{V \otimes \dots \otimes V}^{k}$.

fill m-tuples with all possible partitions of k. (will have multiplicities)

<u>Ex</u>: $\Lambda^2 \mathbb{C}^m$ $\quad e_i \wedge e_j \to (0\,\overset{i}{1}\,0 \dots 0\,\overset{j}{1}\,0)$

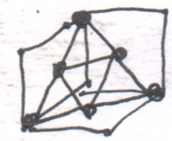
(all mult 1)

$m=3$: $(1\,1\,0)(1\,0\,1)(0\,1\,1)$ →

<u>Ex</u>: $\mathrm{Sym}^2 \mathbb{C}^m$ $\quad e_i e_j \quad (0\,\overset{i}{1} \dots \overset{j}{1}\,0)$ (mult 1)

$e_i^2 \quad (0 \dots 2 \quad 0)$ (mult 1)

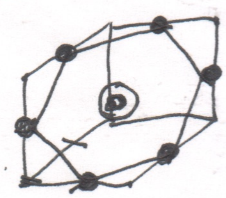
$m=3$: $(1\,1\,0)(1\,0\,1)(0\,1\,1)$
$(2\,0\,0)(0\,2\,0)(0\,0\,2)$



<u>Ex</u>: $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$ $\mathbb{C}^3$

11 2	12 2	11 3	13 3	23 3
22 3	13 2	12 3		

$(2\,1\,0)(1\,2\,0)(2\,0\,1)(1\,0\,2)(0\,1\,2)$
$(0\,2\,1)(1\,1\,1)(1\,1\,1)$



<u>General Rule for E^λ:</u>

① m-tuples: sum to $|\lambda|$

② filling → $(\#1s, \#2s, \#3s, \dots, \#ms)$

<u>Recall</u>: to each filling is basis element

$e_T + I$ in $E^\lambda = V \otimes \dots \otimes V / I$

<u>Ex</u>: $\begin{array}{|c|c|}\hline 1&1\\ \hline 2 \\ \cline{1-1}\end{array} \to (e_1 \wedge e_2) \otimes e_1 + I$

$\pi(t) \begin{array}{|c|c|}\hline 1&1\\ \hline 2 \\ \cline{1-1}\end{array} = (t_1^2 t_2) \begin{array}{|c|c|}\hline 1&1\\ \hline 2 \\ \cline{1-1}\end{array}$

11/22/2016

$G = GL(m, \mathbb{C}) \circlearrowright \mathbb{C}^m$

$T = \text{diag} = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}$

(π, V) finite dim'l rep'n, (V fin dim'l v.s. / $\mathbb{C}$)

$\pi : G \underset{\text{hom}}{\to} GL(V)$

$\pi(gh) = \pi(g)\pi(h)$

Restrict π to T:

$\pi(t)$ are simultaneously diagonalizable.

$\underset{\neq 0}{v} \in V$ is called a <u>weight</u> <u>vector</u> of <u>weight</u> $k = (k_1, \ldots, k_m)$

if $\pi(t)v = t_1^{k_1} \cdots t_m^{k_m} v$

(joint eigenvector for $\pi(t)$)
eigenvalue

Tableau / Diagram λ

$|\lambda| = n$ boxes, $\sum k_i = n$

$E^\lambda = \underbrace{V \otimes \cdots \otimes V}_{n} / I$

Basis: each filling $\underset{\substack{T \\ w/1,\ldots,m}}{} \to e_T + I$

$\boxed{\text{weight is } (\#1\text{s}, \#2\text{s}, \ldots, \#m\text{s})} \boxed{= k_T}$

Ex: $\mathbb{C}^3$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} + I = (e_1 \wedge e_2) \otimes e_2 + I$

$\pi\left(\begin{smallmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{smallmatrix}\right)\left[(e_1 \wedge e_2) \otimes e_2\right] = t_1^2 t_2^1 \left[(e_1 \wedge e_2) \otimes e_2\right] + I$

$\to$ weight $\underset{k_1\ k_2\ k_3}{(2, 1, 0)}$

$\pi(t_1 t_2 t_3) e_1 = t_1 e_1$ $\quad \pi(t_1 t_2 t_3) e_3 = t_3 e_3$

$\pi(t_1 t_2 t_3) e_2 = t_2 e_2$ $\quad \pi(g)(v_1 \otimes v_2) = g v_1 \otimes g v_2$

Two features of weight diagrams:

1. S_m-symmetry (Weyl group)
2. Raising / Lowering Operators (Borel subgroup)

(1) Permutation Matrices $\subseteq GL(m, \mathbb{C})$

P is a <u>permutation matrix</u> if exactly one 1 in each row column.

That is, just permute columns of identity matrix. I_m.

Bijection: $S_m \to \text{Perm}(m)$

$\sigma \to \underset{P_\sigma =}{\left[e_{\sigma(1)} \cdots e_{\sigma(m)}\right]}$

$e_i \mapsto e_{\sigma(i)}$

homomorphism:

$\tau \circ \sigma \mapsto P_\tau \circ P_\sigma$

<u>Note</u>: Perm(m) group under matrix mult

1. $P_\sigma^{-1} = P_\sigma^T$ (orthogonal)
2. Each P_σ has $P_\sigma^j = I$ for some j

$\boxed{\text{Note: Using Perm}(m), \ S_m \text{ acts on the weight set of any } (\pi, V) \text{ on } GL(m, \mathbb{C})}$

This accounts for six fold symmetry in $GL(3, \mathbb{C})$ examples

$\to S_3$ action $\to$ dihedral group

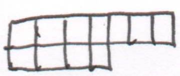
$\boxed{\text{Proposition: If } v \text{ in } V \text{ is a weight vector of weight } k=(k_1 \ldots k_m) \text{ then } P_\sigma v \text{ is a weight vector of weight } \sigma k = (k_{\sigma^{-1}(1)}, \ldots, k_{\sigma^{-1}(m)})}$

Proof: $P_\sigma v$

$$t P_\sigma v = P_\sigma \underbrace{P_\sigma^{-1} t P_\sigma}_{\begin{bmatrix} t_{\sigma(1)} & & \\ & \ddots & \\ & & t_{\sigma(m)} \end{bmatrix}} v = t_{\sigma(1)}^{k_1} \cdots t_{\sigma(m)}^{k_m} P_\sigma v$$

$$= t_1^{k_{\sigma^{-1}(1)}} \cdots t_m^{k_{\sigma^{-1}(m)}} P_\sigma v \;//$$

Ex: $G = GL(2, \mathbb{C})$

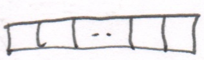
General:  λ_1 λ_2

$\Lambda^2 \mathbb{C}^2 \simeq \det \to$ first λ_2 cols $\to \det^{\lambda_2}$

(λ_2, λ_2)

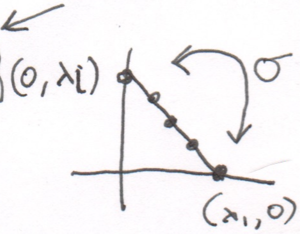
$\det^k(t_1 t_2) = t_1^k t_2^k.$

(polys in e_1, e_2)

 $\lambda_1 \to \mathrm{Sym}^{\lambda_1}(\mathbb{C}^2)$

Basis: $e_1^i e_2^{\lambda_1 - i}$

$\boxed{\text{weights: } (i, \lambda_1 - i)}$



$W = S_2 = \mathbb{Z}/2$

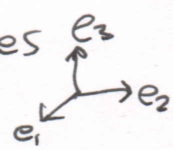
$\sigma : (i, \lambda_1 - i) \longmapsto (\lambda_1 - i, i)$

$e_1 \to e_2$

$e_2 \to e_1$

Ex: $G = GL(3, \mathbb{C})$

Each σ permutes the axes of $\mathbb{R}^3$



② Raising / Lowering Operators

Restrict to T: lose cohesion due to $GL(m, \mathbb{C})$

To repair, raise/lower weights

$GL(2, \mathbb{C})$

$$E^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

How to differentiate a representation:

Directional derivative (Calc III)

$$(Xf)(a) = \frac{d}{dt} f(c(t)) \Big|_{t=0}$$

where $c(t)$ smooth curve in space with $c(0) = a$, $c'(0) = X$

M any matrix $m \times m / \mathbb{C}$

$$\exp(M) = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$$

always convergent.

Ex: $\exp\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + 0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$$\exp\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

$$\exp\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \quad \exp\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}$$

$$\exp\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

11/22/2016

$\mathbb{C}^2$

$$E^+ e_1 = \frac{d}{dt}\, \pi(\exp tE^+)\, e_1 \Big|_{t=0}$$

$$= \frac{d}{dt} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e_1 \Big|_{t=0} = 0$$

$E^+ e_2 = e_1$

$E^- e_1 = e_2$

$E^- e_2 = 0$

$H_1 e_1 = e_1 \qquad H_2 e_1 = 0$

$H_1 e_2 = 0 \qquad H_2 e_2 = e_2$

E^+, E^-, H_1, H_2 act on tensors using the product rule

$$E^+(v_1 \otimes \dots \otimes v_k) = E^+ v_1 \otimes \dots \otimes v_k + v_1 \otimes E^+ v_2 \otimes \dots \otimes v_k + \dots$$

On $\mathrm{Sym}^\lambda(\mathbb{C}^2)$ $\qquad \boxed{\ \ \big|\ \ \big|\ \ \big|\ \ \big|\ \ }$

Basis: $e_1^i e_2^{\lambda-i} \Rightarrow (i, \lambda-i)$

$E^+ : e_1^i e_2^{\lambda-i} \to e_1^{i+1} e_2^{\lambda-i-1} \quad (\lambda-i)$

$E^- : e_1^i e_2^{\lambda-i} \to e_1^{i-1} e_2^{\lambda-i+1} \quad (i)$

$H_1 : e_1^i e_2^{\lambda-i} \to i\, e_1^i e_2^{\lambda-i}$

$H_2 : e_1^i e_2^{\lambda-i} \to \lambda-i\, e_1^i e_2^{\lambda-i}$

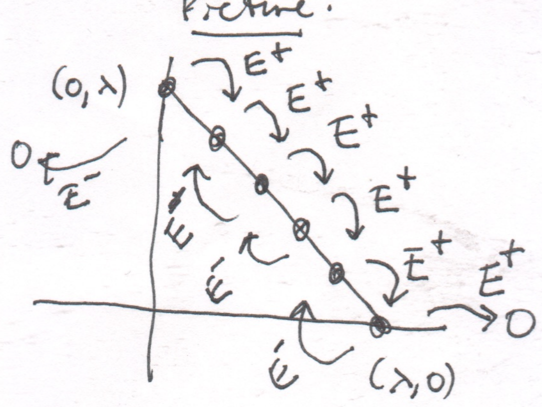
records exponents $\to$ weights

weights:

$(i, \lambda-i) \to (i+1, \lambda-i+1)$

$(i, \lambda-i) \to (i-1, \lambda-i+1)$

Picture:



11/29/2016

$G = GL(m, \mathbb{C})$

(π, V) rep'n

$\longrightarrow$ linearize by differentiating.

$\mathfrak{g} = M_m(\mathbb{C}) = \{m \times m \text{ matrices over } \mathbb{C}\}$

$A \in M_n(\mathbb{C})$

$\exp(A) = I + A + \frac{A^2}{2!} + \cdots$

always converges

$v \in V$: $A \cdot v = \frac{d}{dt} \exp(tA) v \big|_{t=0}$

$V = \mathbb{C}^n$: $A \cdot v = Av$ (mtx-vector product)

$E^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$: $e_1 \to 0$, $e_2 \to e_1$ $\qquad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $e_1 \to e_1$, $e_2 \to 0$

$E^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$: $e_1 \to e_2$, $e_2 \to 0$ $\qquad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $e_1 \to 0$, $e_2 \to e_2$

For Schur modules, extend action on $\mathbb{C}^m$ using product rule on tensors.

$A(v \otimes w) = Av \otimes w + v \otimes Aw$, etc.

Want: associate algebra for ring theory

Step 1: Lie algebra $\mathfrak{g} = M_m(\mathbb{C})$

bracket $[X, Y] = XY - YX$

1. $[\cdot, \cdot]$ bilinear
2. $[X, Y] = -[Y, X]$
3. (Jacobi identity) $[X[YZ]] + [Y[ZX]] + [Z[XY]] = 0$

Note: $[X, Y]$ measures defect in commutativity

1. $XY = YX + [X, Y]$

2. Not associative in general

$$[X[Y, Z]] = [[XY]Z] + \underbrace{[Y[X, Z]]}_{\text{defect}}$$

To get associative algebra

$\longrightarrow$ universal enveloping algebra $U(\mathfrak{g})$

Tensor Algebra

$T_n = V \otimes \cdots \otimes V$ (n times)

$T = \bigcup_n T_n V$ $\to$ ring: mult is concatenation of monomials.

$Sym(V) = T / \langle x \otimes y - y \otimes x \rangle$ homogen. ideal

$\Lambda(V) = T / \langle x \otimes y + y \otimes x \rangle$ homog. ideal.

$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle X \otimes Y - Y \otimes X - [X, Y] \rangle$

not homog!

$U(\mathfrak{g})$
1. associative
2. not nec. commutative

$X \cdot Y = Y \cdot X + [X, Y]$ not matrix mult!

Point: this lets us repeatedly apply elements of Lie algebra ($M_m(\mathbb{C})$) in consistent ring theoretic setting.

(We use without mention!)

Return to Weight Diagrams: irreducible (π, V) $V = E^\lambda$

① W = permutation matrices $\rightarrow$ all weights in convex hull of $W \cdot$ (highest wt)

② To glue, repeatedly apply elements of E_{ij} $(i>j)$ to highest weight vector.

$GL(3,\mathbb{C})$: $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}$ hw $(2,2,0)$ $\rightarrow$ ① $W\cdot(2,2,0) = \{(2,2,0),(2,0,2),(0,2,2)\}$

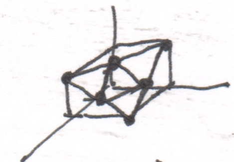
$m=3$

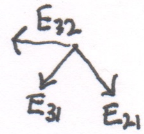
$V=\mathbb{C}^3$

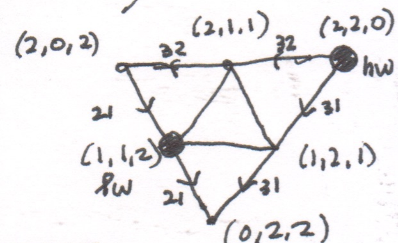
② E_{21} : $e_1\to e_2$, $e_2\to 0$, $e_3 \to 0$	E_{31} : $e_1 \to e_3$, $e_2\to 0$, $e_3\to 0$	E_{32} : $e_1\to 0$, $e_2\to e_3$, $e_3\to 0$
$+1$ to 2^{nd} coord, -1 to 1^{st} coord	$+1$ to 3^{rd}, -1 to 1^{st}	$+1$ for 3^{rd}, -1 to 2^{nd}

Effect on weights. If not $\to 0$ $\Longrightarrow$

$(2,2,0)$	$(0,2,2)$	$(1,2,1)$	$(2,0,2)$	$(2,1,1)$	$(1,1,2)$
$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&2\\\hline 3&3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 2&3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 3&3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&3\\\hline\end{array}$



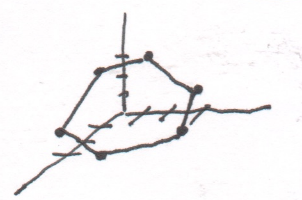




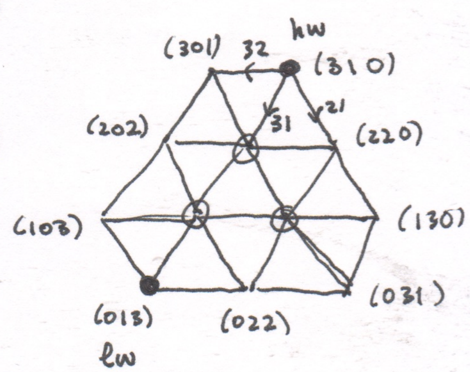
$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}$

$(3,1,0)$ hw

$W\cdot(3,1,0) = \left\{\begin{array}{ccc}(3,1,0) & (1,3,0) & (0,1,3)\\ (3,0,1) & (1,0,3) & (0,3,1)\end{array}\right\}$



(310)	(301)	(220)	(211)
$\begin{array}{\|c\|c\|c\|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&1\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&2\\\hline 3\\\cline{1-1}\end{array}$
(211)	(202)	(130)	(121)
$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&2\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&2\\\hline 3\\\cline{1-1}\end{array}$
(121)	(112)	(112)	(103)
$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&3\\\hline 3\\\cline{1-1}\end{array}$
(022)	(013)		
$\begin{array}{\|c\|c\|c\|}\hline 2&2&3\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 2&3&3\\\hline 3\\\cline{1-1}\end{array}$		



12/6/2016

$(\pi, V), (\pi', V')$ fin dim'l reps of $GL(m,\mathbb{C}) = G$ over $\mathbb{C}$

Definition: $L : V \to V'$ is called an intertwining operator (or G-map, or G-equivariant) if $\pi'(g) L = L \pi(g)$ for all $g \in G$.

($\Rightarrow$ morphism of rep'ns)

Definition: If L is iso. of vector spaces, then L is called an equivalence.

($\Rightarrow$ isomorphism of rep'ns)

Rather than construct L every time, would rather have a classification.

$\to$ "Theorem of the Highest Weight"

For polynomial rep'ns of $GL(m,\mathbb{C})$

$\exists$ 1-1 correspondence

$$\left\{\begin{array}{c}\text{irred poly repns}\\ \text{of } GL(m,\mathbb{C})\end{array}\right\} \simeq \left\{\begin{array}{l}\text{diagrams,}\\ \text{w/ tableau}\\ \text{in } 1,\dots,m\end{array}\right\}$$

mod equivalence

E^λ Schur module $\longleftarrow$ diagram λ

Refine: use diagram w/ $< m$ rows

$$\det^k \otimes E^\lambda \longleftarrow \left\{\begin{array}{c}\text{diagram}\\ < m \text{ rows}\end{array}\right\} \times \mathbb{Z}_{\geq 0}$$

Recall highest wt: $\begin{array}{|c|c|c|}\hline 1 & 1 & 1\\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \to (3,2,1,0,\dots)$

Also:

$\exists$ 1-1 correspondence between

$$\left\{\begin{array}{c}\text{irred fin dim'l}\\ \text{rep'ns of } GL(m,\mathbb{C})\end{array}\right\} \simeq \left\{\begin{array}{c}\text{diagrams}\\ < m \text{ rows}\end{array}\right\} \times \mathbb{Z}$$

mod equivalence

$$\det^k \otimes E^\lambda \longleftarrow (\text{diagram } \lambda, k)$$

Example: $GL(2,\mathbb{C})$

$Sym^k(\mathbb{C}^2)$ = homog polys in e_1, e_2

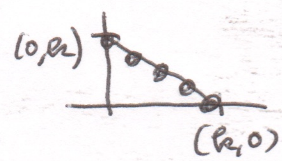
basis of weight vectors $\{e_1^{k-i} e_2^{i}\}$

$(k-i, i) \to$ highest $(k,0)$

$\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline\end{array}$

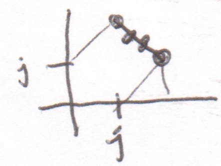
k boxes

dim $= k+1$



$\det^j \otimes Sym^k(\mathbb{C}^2)$

(j,j) + weight $(k-i, i)$



Problem #1: Clebsch-Gordan.

Consider $Sym^2(\mathbb{C}^2) \otimes Sym^3(\mathbb{C}^2)$

$GL(2,\mathbb{C})$ acts by $g(v \otimes w) = gv \otimes gw$.

So on weight vectors: of wt λ_1, λ_2

$$\begin{aligned} t(v_1 \otimes v_2) &= tv_1 \otimes tv_2 \\ &= t^{\lambda_1} v_1 \otimes t^{\lambda_2} v_2 \\ &= t^{\lambda_1 + \lambda_2}(v_1 \otimes v_2) \end{aligned}$$

New weights: sums of old weights pairwise.

Sym^2	Sym^3	$\to$			
			(5,0)	(4,1)	(3,2)
(2,0)	(3,0)	sum	(4,1)	(3,2)	(2,3)
(1,1)	(2,1)		(3,2)	(2,3)	(1,4)
(0,2)	(1,2)		(2,3)	(1,4)	(0,5)
	(0,3)				

12/6/2016 ②

Note: Equivalence preserves weight sets

$$\pi'(t)Lv = L\pi(t)v = Lt^\lambda v = t^\lambda Lv$$

so v, Lv have weight λ. //

Picture of $Sym^2 \otimes Sym^3$

	highest weight
$\simeq Sym^5(\mathbb{C}^2)$	$(5,0)$
$\oplus \det^1 \otimes Sym^3(\mathbb{C}^2)$	$(4,1)$
$\oplus \det^2 \otimes Sym^1(\mathbb{C}^2)$	$(3,2)$

Note: $\det^k \to$ add (k,k) ↗ so off x,y-axes.

Problem #2: How to realize E^λ as vector space of polynomials (for algebraic geometry)

$\mathbb{C}[z_{ij}]$ $\quad 1 \le i \le n \quad (=|\lambda|)$

$\qquad 1 \le j \le m$

Recipe:
⓪ Fix tableau T
① Factor for each column of T

$$\to \boxed{D_T = \prod_{cols} \det(M_{col})}$$

② Each entry in col of T $\to$ col in matrix $[z_{ij}]$
③ Arrange in order $\to$ chop bottom off $\to$ square
④ det square matrix

Example: $\lambda = \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ $\quad \mathbb{C}^3$ $\quad m=3$, $n=2$, $GL(3,\mathbb{C})$

$$\to [z_{ij}] = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{bmatrix}$$

$\begin{array}{cc} c_1 & c_1 \\ \hline 1 & 1 \end{array} \to \det(z_{11})\det(z_{11}) = z_{11}^2$

$\begin{array}{cc} 2 & 3 \\ \hline c_2 & c_3 \end{array} \to \det(z_{12})\det(z_{13}) = z_{12}z_{13}$

$\to$ Basis: $\{z_{11}^2, z_{12}^2, z_{13}^2, z_{12}z_{13}, z_{11}z_{12}, z_{11}z_{13}\}$

Ignore 1 $\to Sym^2(\mathbb{C}^3)$

$GL(3,\mathbb{C})$ action: $[g \cdot P](A) = P(Ag)$

P poly in $\longrightarrow$ $n \times m$ variables

Example: $\lambda = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\quad \mathbb{C}^3$ $\quad m=3$, $n=3$

$$\to [z_{ij}] = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$$

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \det\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{bmatrix}\det(z_{11})$

$= z_{11}(z_{11}z_{22} - z_{12}z_{21})$

$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \det\begin{bmatrix} z_{12} & z_{13} \\ z_{22} & z_{23}\end{bmatrix}\det(z_{12})$

$= z_{12}(z_{12}z_{23} - z_{22}z_{23})$

In E^λ: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1$

$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to e_2 \otimes e_3 \otimes e_2 - e_3 \otimes e_2 \otimes e_2$

Check: D_T satisfies 3 props:
① multilinear in cols of z_{ij}
② alternating in cols of z_{ij}
③ $D_T(M) = \sum_i D_T(M')$ (Sylvester)