

# Counting formulas for anti-magic squares and conjoined compositions

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May 1, 2026

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# Binomial Coefficient as Polynomial in $t$ for fixed $n$

$$\binom{t+n}{n} = \frac{(t+n)!}{n! t!} = \frac{(t+n)(t+n-1)\dots(t+2)(t+1)}{n!}$$

When  $t$  is a non-negative integer:

- Number of ways to put  $t$  balls into  $n + 1$  boxes
- Number of **weak compositions** of  $t$  with  $n + 1$  parts
- (Number of solutions to  $x_1 + x_2 + \dots + x_{n+1} = L$  with  $x_i \geq 0$ )
- (Number of integer points in the  $n$ -simplex, dilated by a factor of  $L$ )

**Weak composition:**  $(n + 1)$ -tuple of non-negative integers that sum to  $t$

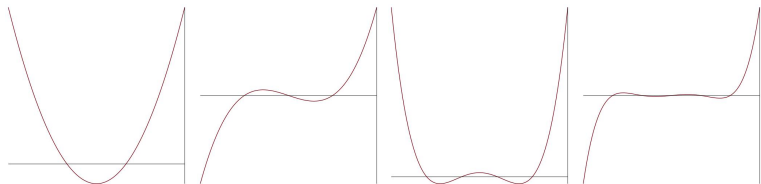
2	2	0	3	5
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is a weak composition of 12 with 5 parts

# Binomial Coefficient as Polynomial in $t$

$$Q_n(t) = \frac{(t+n)(t+n-1)\dots(t+2)(t+1)}{n!}$$

- $Q_n(0) = 1$
- $Q_n(t) = 0$  for  $t = -1, -2, \dots, -n$
- Odd/evenness of graphs about the axis  $x = -\frac{n+1}{2}$



# Reciprocity for Binomial Coefficients

Replace  $t$  with  $-t$ :

$$\boxed{\binom{-t+n}{n} = (-1)^n \binom{t-1}{n}}$$

*Proof:*

$$\begin{aligned} \binom{-t+n}{n} &= \frac{(-t+n)(-t+n-1)\dots(-t+2)(-t+1)}{n!} \\ &= (-1)^n \frac{(t-n)(t-n+1)\dots(t-2)(t-1)}{n!} \\ &= (-1)^n \binom{t-1}{n}. \quad QED \end{aligned}$$

# Reciprocity for Binomial Coefficients

$$\boxed{\binom{-t+n}{n} = (-1)^n \binom{(t-n-1)+n}{n}}$$

## Interpret using combinatorial reciprocity:

Recall that the lefthand side is the number of weak compositions of  $t$  with  $n + 1$  parts.

Up to sign, the righthand side is the number of (strict) compositions of  $t$  with  $n + 1$  parts. In this case, only positive integers are allowed.

To see this, we first place a 1 in each of the  $n + 1$  boxes, reducing the available number of balls to  $t - n - 1$ .

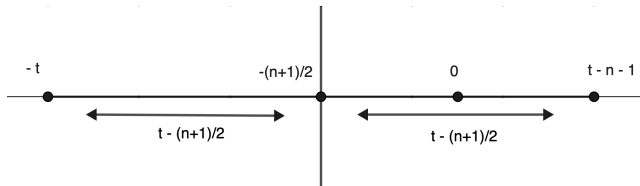
# Reciprocity for Binomial Coefficients

$$\binom{-t+n}{n} = (-1)^n \binom{(t-n-1)+n}{n}$$

$$Q_n(-t) = (-1)^n Q_n(t - n - 1)$$

**Interpret as reflection:**

$$\text{Midpoint } (-t, t - n - 1) = -\frac{n+1}{2}$$



# Reciprocity for Binomial Coefficients

$$Q_n(t) = \frac{(t+n)(t+n-1)\dots(t+2)(t+1)}{n!}$$

**Interpret as even/odd polynomial:**

$Q_n(t)$  is even/odd as a function of  $t + \frac{n+1}{2}$  or  $2t + n + 1$

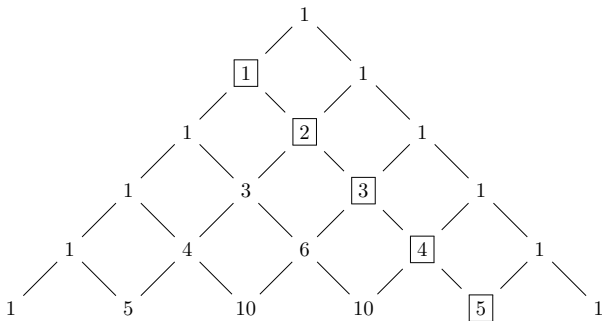
- $Q_0(t) = 1$
- $Q_1(t) = t + 1$
- $Q_2(t) = \frac{1}{2!}(t^2 + 3t + 2) = \frac{1}{8}[(2t + 3)^2 - 1]$
- $Q_3(t) = \frac{1}{3!}(t + 2)[(t + 2)^2 - 1]$
- ...
- $Q_{2m-1}(t) = \frac{1}{(2m-1)!}(t + m)[(t + m)^2 - 1^2] \dots [(t + m)^2 - (m - 1)^2]$

# Binomial Series and Pascal's Triangle

Binomial series:

$$\frac{1}{(1-x)^{n+1}} = 1 + \binom{n+1}{1}x + \binom{n+2}{2}x^2 + \cdots + \binom{n+k}{k}x^k + \cdots$$

$n = 1 :$



# Binomial Series and Pascal's Triangle

Geometric series:

$$\frac{1}{(1-x)} = 1 + x + x^2 + \dots$$

Generating function for the sequence  $a_k$ :

$$F(x) = a_0 + a_1x + a_2x^2 + \dots$$

Then  $\frac{1}{1-x}F(x)$  is the generating function for the sequence of partial sums

$$s_k = a_0 + a_1 + \dots + a_k$$

In turn,  $\frac{1}{(1-x)^{n+1}}F(x)$  implements  $n + 1$  partial sum operations.

If  $F(x) = 1$ , we obtain the binomial series and hyperpyramidal figurate numbers (triangular numbers, tetrahedral numbers, etc.)

# Binomial Series as Iterated Partial Sums

To the right: multiply by  $\frac{1}{1-x}$

Partial Sum / Hockey Stick Rule:

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

Pascal's Identity:

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

# Vandermonde Convolution

$$\sum_{i+j=k} \binom{m}{i} \binom{n}{j} = \binom{m+n}{k}$$

*Proof 1:* Disjoint union  $X \cup Y$  with  $|X| = m$ ,  $|Y| = n$ .

To choose  $k$  elements from  $X \cup Y$ ,  
first choose  $i$  from  $X$ , then  $k - i$  from  $Y$ .

Then sum over  $i$ . QED

*Proof 2:* Match the coefficients of  $x^k$  in

$$(1 + x)^m (1 + x)^n = (1 + x)^{m+n}.$$

QED

## Vandermonde Convolution

1) VDC is a family of identities for fixed  $m + n$  and  $k$ :

$(m, n), (m - 1, n + 1), (m - 2, n + 2), \dots$

$$(1 + x)^m (1 + x)^n = (1 + x)^{m-1} (1 + x)^{n+1} = \dots = (1 + x)^{m+n}$$

2) Bilinear form on power series: fix  $k$

$$\langle p(x), q(x) \rangle_k = [x^k] p(x) q(x)$$

$$\langle p(x), q(x) \rangle_k = p_0 q_k + p_1 q_{k-1} + \dots + p_k q_0$$

(Reverse one set of coefficients, then dot product)

Invariance under multiplication:

$$\langle h(x)p(x), q(x) \rangle_k = \langle p(x), h(x)q(x) \rangle_k$$

Special:  $h(x) = \frac{1}{1-x}$  is the iterating factor for the table

# Vandermonde Convolution (Partial Sums)

$$1 * 10 + 0 * 6 + 0 * 3 + 0 * 1 = 10$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

$$1 * 4 + 1 * 3 + 1 * 2 + 1 * 1 = 10$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

$$1 * 20 + (-1) * 10 + 0 * 4 + 0 * 1 = 10$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

# Semi-magic squares

A square matrix is called a **semi-magic square** if

- 1 entries are integers  $\geq 0$ , and
- 2 the sum along any row or column is equal to the same number  $L$ .

$L$  is called the **line sum** of  $M$ .

**Examples:**

$$\begin{bmatrix} 3 & 1 & 2 \\ \boxed{2} & \boxed{2} & \boxed{2} \\ 1 & 3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} \end{bmatrix}, \quad \begin{bmatrix} 1 & \boxed{2} & 3 & 4 \\ 2 & \boxed{3} & 4 & 1 \\ 3 & \boxed{4} & 1 & 2 \\ 4 & \boxed{1} & 2 & 3 \end{bmatrix}.$$

## Some Highlights:

- P. McMahon (1916): Counting formula for size 3
- Coupling of angular momentum in particle physics (late 1920s)
- Birkhoff-Von Neumann Theorem (1946)
- Anand-Dumir-Gupta Conjecture (1966), extended and proved by R. Stanley
- M. Beck, D. Pixton (2003): Counting formula for size 9
- J. De Loera, F. Liu, R. Yoshida (2009): Generating function for general case

# Example: Permutation Matrices

Size 3,  $L = 1$

Rotations:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Reflections:

$$P_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

## Example: Sums of Permutation Matrices

Any linear combination with  $a_i \geq 0$  integers

$$a_1 P_1 + \dots + a_6 P_6$$

is a semi-magic square with line sum  $L = a_1 + \dots + a_6$ .

### Corollary of Birkhoff-Von Neumann Theorem

Every semi-magic square with line sum  $L$  is a sum of  $L$  permutation matrices.

**Difficulty in general:** linear dependence and relations over  $\mathbb{N}$

## Size 3: Unique relation

Solve:

$$\sum a_i P_i = \begin{bmatrix} a_1 + a_6 & a_3 + a_5 & a_2 + a_4 \\ a_2 + a_5 & a_1 + a_4 & a_3 + a_6 \\ a_3 + a_4 & a_2 + a_6 & a_1 + a_5 \end{bmatrix} = 0.$$

Solution:

$$a_1 = a_2 = a_3 = 1, \quad a_4 = a_5 = a_6 = -1,$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

### Consequences:

- 1) If some entry of  $M$  equals zero, then the sum is unique.
- 2) Otherwise, there are  $m + 1$  distinct sums that represent  $M$ , where  $m$  is the smallest entry of  $M$ .

# Rectangular notation

Every semi-magic square  $M$  can be represented by a rectangle:

$$M \leftrightarrow \mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \leftrightarrow \begin{array}{|c|c|c|} \hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline \end{array}.$$

Here the single relation takes the form

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array}.$$

By repeatedly shifting up, uniquely represented if one of  $a_4, a_5, a_6$  is zero.

Note: line sum  $L = a_1 + \cdots + a_6$  is unchanged by shifting 1s

# Anti-magic squares

Size 4 and beyond: Too many relations

(Anand-Dumir-Gupta Conjecture, extended and proved by R. Stanley)

Alternative model that generalizes the size 3 case:

**Anti-magic squares** (R. Stanley, EC1, Ch. 4, Problem 53)

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

## Example of Anti-magic Square

$$\begin{aligned}
 M &= (3C_1 + 4C_2 + 2C_3) + (0R_1 + 1R_2 + 2R_3) \\
 &= (2C_1 + 3C_2 + 1C_3) + (1R_1 + 2R_2 + 3R_3) \\
 &= (1C_1 + 2C_2 + 0C_3) + (2R_1 + 3R_2 + 4R_3)
 \end{aligned}$$

$$i(M) = 2 + 3 + 1 + 1 + 2 + 3 = 12$$

$$\begin{array}{ccc}
 \begin{bmatrix} \boxed{3} & 4 & 2 \\ 4 & \boxed{5} & 3 \\ 5 & 6 & \boxed{4} \end{bmatrix}, & \begin{bmatrix} 3 & 4 & \boxed{2} \\ \boxed{4} & 5 & 3 \\ 5 & \boxed{6} & 4 \end{bmatrix}, & \begin{bmatrix} 3 & \boxed{4} & 2 \\ 4 & 5 & \boxed{3} \\ \boxed{5} & 6 & 4 \end{bmatrix} \\
 \\
 \begin{bmatrix} 3 & 4 & \boxed{2} \\ 4 & \boxed{5} & 3 \\ \boxed{5} & 6 & 4 \end{bmatrix}, & \begin{bmatrix} 3 & \boxed{4} & 2 \\ \boxed{4} & 5 & 3 \\ 5 & 6 & \boxed{4} \end{bmatrix}, & \begin{bmatrix} \boxed{3} & 4 & 2 \\ 4 & 5 & \boxed{3} \\ 5 & \boxed{6} & 4 \end{bmatrix}
 \end{array}$$

# Rectangular notation

For size  $n$ , we obtain a  $2 \times n$  rectangle by

$$M = c_1 C_1 + \cdots + c_n C_n + r_1 R_1 + \cdots + r_n R_n$$

$$\rightarrow (c_1, \dots, c_n, r_1, \dots, r_n)$$

$$\rightarrow \begin{array}{|c|c|c|c|} \hline c_1 & c_2 & \dots & c_n \\ \hline r_1 & r_2 & \dots & r_n \\ \hline \end{array}$$

with the single relation

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & \dots & 1 \\ \hline 0 & 0 & \dots & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & \dots & 0 \\ \hline 1 & 1 & \dots & 1 \\ \hline \end{array}.$$

## Counting Anti-magic Squares (Exercise 53)

**Upshifted rectangles.** At least one zero in the bottom row.

Let  $i(M) = \sum r_i + \sum c_i$  be the **index** of  $M$ ,

and  $\rho_n(L)$  the number of anti-magic squares of size  $n$  and index  $L$ .

$$\rho_n(L) = \binom{L+2n-1}{2n-1} - \binom{L+n-1}{2n-1}$$

*Proof:* The first term is the number of weak compositions (rectangles) of  $L$  with  $2n$  parts.

The second term is the number such rectangles with non-vanishing parts in the bottom row.

(Pre-assign a ball to each box in the bottom row) QED

$$F_n(x) = \frac{1-x^n}{(1-x)^{2n}} = \frac{1+x+\dots+x^{n-1}}{(1-x)^{2n-1}}$$

# Conjoined Compositions

Extend to  $m \times n$  rectangles:

**Definition:**

A **conjoined composition of type  $(m, n)$  and weight  $L$**  is a weak composition of  $L$  with  $mn$  parts subject to the shift relation.

**Example:** Type  $(3, 4)$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} .$$

## Explicit model

Generating set as  $mn$ -tuples

Type (2, 3):

110000, 001100, 000011

011000, 000110, 100001

Type (3, 2):

111000, 000111

110001, 001110

100011, 011100

**General:** rows of circulant semi-magic square

1100

0110

0011

1001

# Main Result:

Theorem: (QED REU 2026)

Let  $\rho_{m,n}(L)$  be the number of conjoined compositions of type  $(m, n)$  with weight  $L$ . Then the generating function for  $\rho_{m,n}(L)$  is given by

$$F_{m,n}(x) = \sum_{L=0}^{\infty} \rho_{m,n}(L)x^L = \frac{(1-x^n)^{m-1}}{(1-x)^{mn}}.$$

Anti-magic squares:

$$F_{2,n}(x) = \frac{1-x^n}{(1-x)^{2n}}.$$

## Proof by induction

We count rectangles in upshifted form.

**Base cases:** Weak compositions / Anti-magic squares

**Induction step:** Assume the formula is true for CCs of type  $(m, n)$ .

Type  $(m + 1, n)$ :  $F_{m+1,n}(x)$

Type  $(m, n)$ :  $F_{m,n}(x)$

Bottom row: a weak composition with  $n$  parts and at least one zero.

$$\rho(i) = \binom{i+n-1}{n-1} - \binom{i-1}{n-1} \quad \rightarrow \quad f(x) = \frac{1-x^n}{(1-x)^n}$$

Two independent choices: multiply the generating functions

$$F_{m+1,n}(x) = \frac{1-x^n}{(1-x)^n} F_{m,n}(x)$$

QED

# Counting formulas: Anti-magic squares

$$\rho_n(L) = \binom{L+2n-1}{2n-1} - \binom{L+n-1}{2n-1}$$

Generating function:

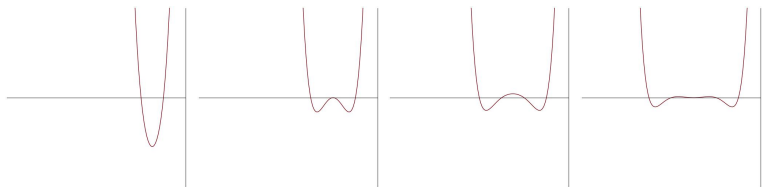
$$F_n(x) = \frac{1-x^n}{(1-x)^{2n}} = \frac{1+x+x^2+\dots+x^{n-1}}{(1-x)^{2n-1}}$$

- $\rho_n(L)$  is a polynomial of degree  $2n - 2$
- $\rho_n(L) = 0$  for  $L = -1, -2, \dots, -(n - 1)$
- $\rho_n(-L) = \rho(L - n)$

# Counting formulas

$$\rho_n(-L) = \rho_n(L - n)$$

$\rho_n(L)$  is an even polynomial in  $L + \frac{n}{2}$



# Even as a Polynomial in $L + m$

$n = 2m :$

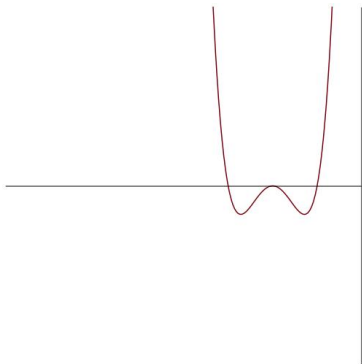
$$\begin{aligned} \rho_n(L) &= 2/(2n - 1)! \\ &\times (L + m)^2[(L + m)^2 - 1^2] \dots [(L + m)^2 - (m - 1)^2] \\ &\times [e_1(L + m)^{n-2} + e_3(L + m)^{n-4} + \dots + e_{n-1}] \end{aligned}$$

where  $e_k = e_k(m, m + 1, \dots, 3m - 1)$ ,  
the  $k$ -th elementary symmetric function of degree  $2m$

# Even as a Polynomial in $L + 2$

$$\rho_4(L) = 2/7! (L + 2)^2 [(L + 2)^2 - 1^2] [14(L + 2)^2 + 154]$$

$$e_1(2, 3, 4, 5) = 14, \quad e_3(2, 3, 4, 5) = 154$$



## Even as a Polynomial in $L + 2$

### Step 1:

Factor out

$$(L + 1)(L + 2)(L + 3) = ((L + 2) - 1)(L + 2)((L + 2) + 1)$$

### Step 2:

Difference let  $x = L + 2$ :

$$(L + 7)(L + 6)(L + 5)(L + 4) = (x + 5)(x + 4)(x + 3)(x + 2)$$

$$(L - 3)(L - 2)(L - 1)(L) = (x - 5)(x - 4)(x - 3)(x - 2)$$

$$\begin{aligned} (x^4 + e_1x^3 + e_2x^2 + e_1x + e_0) - (x^4 - e_1x^3 + e_2x^2 - e_1x + e_0) \\ = 2e_1x^3 + 2e_3x \end{aligned}$$

# Generating function

$$F_{m,n}(x) = \frac{(1-x^n)^{m-1}}{(1-x)^{mn}} = \frac{(1+x+\dots+x^{n-1})^{m-1}}{(1-x)^{mn-m+1}}$$

- $\rho_{m,n}(L)$  is a polynomial in  $L$  of degree  $m(n-1)$
- $\rho_{m,n}(L) = 0$  for  $L = -1, -2, \dots, -(n-1)$
- $\rho_{m,n}(-L) = (-1)^{m(n-1)} \rho_{m,n}(L-n)$
- $F_{m,n}(x) = \frac{1-x^n}{(1-x)^n} F_{m-1,n}(x)$

# Generating function: $n = 2$

$$F_{m,2}(x) = \frac{(1-x^2)^{m-1}}{(1-x)^{2m}} = \frac{(1+x)^{m-1}}{(1-x)^{m+1}}$$

- $\rho_{m,2}(L)$  is a polynomial in  $L$  of degree  $m$
- $\rho_{m,2}(L) = 0$  for  $L = -1$
- $\rho_{m,2}(-L) = (-1)^m \rho_{m,2}(L-2)$  (symmetric in  $L+1$ )
- $F_{m,2}(x) = \frac{1-x^2}{(1-x)^2} F_{m-1,2}(x) = \frac{1+x}{1-x} F_{m-1,2}(x)$

## Generating function: $n = 2$

Explicit polynomials:

$$\rho_{m,2}(L) = \binom{L+m}{m} + \binom{m-1}{1} \binom{L+m-1}{m} + \binom{m-1}{2} \binom{L+m-2}{m} + \dots$$

$$m = 2 : \rho_{2,2}(L) = 1 \binom{L+2}{2} + 1 \binom{L+1}{2} = (L+1)^2$$

$$m = 3 : \rho_{3,2}(L) = 1 \binom{L+3}{3} + 2 \binom{L+2}{3} + 1 \binom{L+1}{3} = \frac{1}{3}(L+1)[2(L+1)^2 + 1]$$

Factor:

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \dots$$

Hockey stick rule:

$$s_k = a_k + 2a_{k-1} + 2a_{k-2} + 2a_{k-3} + \dots$$

# Iterated Weighted Sums

To the right: multiply by  $\frac{1+x}{1-x}$ . Column zero is  $\frac{1}{1-x^2}$

Hockey Stick Rule Variant:  $44 = 16 + 2(9 + 4 + 1)$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

Pascal's Identity Variant:  $44 = 16 + 19 + 9$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

# Vandermonde Convolution

$$1 * 44 + 0 * 19 + 1 * 6 + 0 * 1 = 50$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

$$1 * 16 + 2 * 9 + 3 * 4 + 4 * 1 = 50$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

$$1 * 96 + (-2) * 33 + 3 * 8 + (-4) * 1 = 50$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

# Generating function: $n = 3$

$$F_{m,3}(x) = \frac{(1-x^3)^{m-1}}{(1-x)^{3m}} = \frac{(1+x+x^2)^{m-1}}{(1-x)^{2m+1}}$$

- $\rho_{m,3}(L)$  is a polynomial in  $L$  of degree  $2m$
- $\rho_{m,3}(L) = 0$  for  $L = -1, -2$
- $\rho_{m,3}(-L) = \rho_{m,3}(L-3)$  (even in  $L+3/2$ )
- $F_{m,3}(x) = \frac{1-x^3}{(1-x)^3} F_{m-1,3}(x) = \frac{1+x+x^2}{(1-x)^2} F_{m-1,3}(x)$

## Generating function: $n = 3$

Explicit polynomials: (use unreduced form of  $F_{m,3}(x)$ )

$$\rho_{m,3}(L) = \binom{L+3m-1}{3m-1} + \dots + (-1)^k \binom{m-1}{k} \binom{L+3m-1-3k}{3m-1} + \dots$$

$$m = 3 : \rho_{3,3}(L) = 1 \binom{L+8}{8} - 2 \binom{L+5}{8} + 1 \binom{L+2}{8}$$

$$= \frac{1}{8! \cdot 16} [(2L+3)^2 - 1][(2L+3)^6 + 421(2L+3)^4 + 25075(2L+3)^2 + 62055]$$

Factor:

$$\frac{1+x+x^2}{(1-x)^2} = 1 + 3x + 6x^2 + 9x^3 + \dots$$

Hockey stick rule:

$$s_k = a_k + 3a_{k-1} + 6a_{k-2} + 9a_{k-3} + \dots$$

# Iterated Weighted Sums

To the right: multiply by  $\frac{1+x+x^2}{(1-x)^2}$ . Column zero is  $\frac{1}{1-x^3}$

Hockey Stick Rule Variant:  $163 = 55 + 3(21) + 6(6) + 9(1)$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

Pascal's Identity Variant:  $163 = 55 + 21 + 6 + 2(45) - 9$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

# Vandermonde Convolution

$$1 * 163 + 0 * 45 + 0 * 9 + 1 * 1 = 164$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

$$1 * 55 + 3 * 21 + 6 * 6 + 10 * 1 = 164$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

$$1 * 361 + (-3) * 78 + 3 * 12 + 1 * 1 = 164$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

## Book References / Thank you!

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