Introduction To
The Borel-Weil-Bott Theorem

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Introduction

Last week:
David Anderson's second talk:

- **Equivariant K-theory for $(G/B, \mathcal{L}(\lambda))$:**
  Weyl character formula as Euler characteristic
- **Borel-Weil theorem:**
  Space of sections on such a line bundle either vanish or realize an irreducible representation of G

Timeline:

- Early 1950s: Borel-Weil Theorem (Serre 1954, Tits 1955)
- 1954: Hirzebruch-Riemann-Roch Theorem
- 1957: Borel-Weil-Bott theorem
- 1958: Grothendieck-Riemann-Roch Theorem (1956-1957)
- 1961: Kostant's Theorem for $\mathfrak{n}$-cohomology

Goal of today's talk:

An explicit formula for harmonic forms using the approach from

- Griffiths-Schmid, Locally homogeneous complex manifolds (1969).

See also Green-Griffiths-Kerr, Hodge theory book, 2013.

Table of contents:

Special thanks to Ari Krishna (Harvard)

Representations of connected compact Lie groups

K connected compact Lie group.

$\mathfrak{k}_0 = Lie(K)$

Definition:

A **representation** (σ, V_σ) is a smooth group homomorphism

$$\sigma : K \to GL(V_\sigma)$$

where V_σ is a finite-dimensional vector space over $\mathbb{C}$.

If we differentiate this action,
we obtain a Lie algebra representation σ of $\mathfrak{k}_0$.

Unitary representations

Definition:

A representation (σ, V_σ) is called **irreducible** if the only invariant subspaces of V_σ are 0 and V_σ.

Definition:

A representation (σ, V_σ) is called **unitary** if there exists an invariant Hermitian inner product on V_σ. That is, for all k in K,

$$\langle \sigma(k)u, \sigma(k)v \rangle = \langle u, v \rangle$$

for all k in K and u, v in V_σ.

If X is in $\mathfrak{k}_0$, we differentiate to get

$$\langle \sigma(X)u, v \rangle = -\langle u, \sigma(X)v \rangle.$$

Full reducibility

Consequence of unitarity:
Every invariant subspace has an invariant orthocomplement.

Full Reducibility

If (π, V_π) is a finite-dimensional representation of K, then

$$V_\pi = V_1 \oplus \cdots \oplus V_n$$

where each (σ_i, V_i) is an irreducible representation of K.

The Peter-Weyl theorem

Peter-Weyl Theorem

Suppose K is a connected compact Lie group with biinvariant Haar measure dk. Then

$$L^2(K) \cong \bigoplus_{\sigma \in \hat{K}} V_\sigma \otimes (V_\sigma)^*$$

as representations of $K \times K$.

Here σ ranges over a set of representatives for each class of irreps of K.

For a fixed σ, we have the **matrix coefficient** map

$$\phi : V_\sigma \otimes (V_\sigma)^* \to L^2(K)$$

$$u \otimes \langle \cdot, v \rangle \mapsto \phi_{u,v}(k) = \langle \sigma(k) u, v \rangle$$

The Peter-Weyl Theorem

Fourier series: If $f \in C^\infty(K) \subset L^2(K)$ then

$$f(k) = \sum_{\substack{\sigma \in \hat{K} \\ i,\ j}} c_{\sigma, i, j} \langle \sigma(k) u_i, u_j \rangle$$

where σ ranges over a set of representatives of irreps of K and $\{u_i\}$ is an orthonormal basis for V_σ.

Group action:

$$\begin{aligned} [(R \otimes L)(k_1, k_2)\phi_{u,v}](k) &= \langle \sigma(k_2^{-1} k k_1) u, v \rangle \\ &= \langle \sigma(k)\sigma(k_1)u, \sigma(k_2)v \rangle \\ &= \phi([\sigma(k_1) \otimes \sigma^*(k_2)] \quad (u \otimes \langle \cdot, v \rangle))(k) \end{aligned}$$

The Peter-Weyl theorem

Lie algebra action (right invariant derivative):

Suppose $X \in \mathfrak{k}_0 = Lie(K)$

$$\begin{aligned} [L(X)\phi_{u,v}](k) &= \tfrac{d}{dt}\langle \sigma(e^{-tX}k)u, v\rangle|_{t=0} \\ &= -\langle \sigma(X)\sigma(k)u, v\rangle \\ &= \langle \sigma(k)u, \sigma(X)v\rangle \end{aligned}$$

Extend to complexification: $Z = X + iY$ in $\mathfrak{k} = \mathfrak{k}_o \otimes \mathbb{C}$

$$\begin{aligned} [L(Z)\phi_{u,v}](k) &= -\langle \sigma(Z)\sigma(k)u, v\rangle \\ &= \langle \sigma(k)u, \sigma(\bar{Z})v\rangle \end{aligned}$$

Useful fact:

On a matrix coefficient for V_σ,
left or right invariant derivatives lie in the same subspace $V_\sigma \otimes (V_\sigma)^*$

Root space decomposition

Suppose K is non-abelian. Then $\mathfrak{k}_0$ is reductive.

Let T be a maximal torus in K,
and define $\mathfrak{t}_0 = Lie(T)$ with $\mathfrak{t} = \mathfrak{t}_0 \otimes \mathbb{C}$.

We have joint eigenspaces for the *ad* action of $\mathfrak{t}$ on

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{k}_\alpha .$$

In this case, $\mathfrak{t}$ is the zero eigenspace.

Otherwise, define the **root space** for the root α

$$\mathfrak{k}_\alpha = \{X \in \mathfrak{k} \mid ad(H)X = [H, X] = \alpha(H)X \text{ for } H \in \mathfrak{t}\},$$

and we denote the set of all **roots** α by Δ.

Root space decomposition

Each root α is a real-valued linear functional on $i\mathfrak{t}_0$.

$B(H_1, H_2) = Tr(ad(H_1), ad(H_2))$ on $\mathfrak{t}$
is real valued and positive definite when restricted to $i\mathfrak{t}_0$.

$\langle \alpha, \beta \rangle = B(H_\alpha, H_\beta)$ where $\langle H_\alpha, H \rangle = \alpha(H)$

Usual notions for root systems Δ:

- Positive roots: Δ^+
- Simple roots: Π
- Weyl group: $W = N_K(T)/T$

For each α in Δ^+, we fix X_α and define $\bar{X}_\alpha = -X_{-\alpha}$.

Check: $\bar{X}_\alpha$ is in $\mathfrak{k}_{-\alpha}$.

Weight space decomposition

Let (π, V) be a finite-dimensional representation of K,

Definition:

The weight space V_μ associated to the weight μ is given by

$$V_\mu = \{v \in V \mid \pi(H)v = \mu(H)v \text{ for } H \in \mathfrak{t}\}$$

Definition:

A weight λ is called **dominant** if $\langle \lambda, \alpha \rangle \geqslant 0$ for all α in Δ^+.

Each irreducible representation σ of K has a unique highest weight λ with respect to Δ^+.

That is, every other weight μ for σ is of the form $\mu = \lambda - \sum_{\alpha \in \Delta^+} n_\alpha \alpha$.

Theorem of the highest weight

The dimension of the highest weight space V_λ is one, and the space is characterized by the property

$$\sigma(X_\alpha)v = 0$$

for all α in Δ^+ and v in V_λ.

If non-zero, such v are called **highest weight vectors**.

Theorem of the Highest Weight

Fix a choice of Δ^+. Then there is a one-one correspondence between equivalence classes of irreducible representations of K and integral dominant weights λ.

Complex variables

Complex coordinate system on $\mathbb{C}^n$: $z_i = x_i + iy_i$, $\bar{z}_i = x_i - iy_i$.

Holomorphic tangent vectors:

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)$$

Anti-holomorphic tangent vectors:

$$\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)$$

Holomorphic functions in several variables

Cauchy-Riemann Equations (one variable):
$F(x, y) = u(x, y) + i\, v(x, y)$:

$$\frac{\partial}{\partial \bar{z}} F(x, y) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)(u + i\, v) = 0$$

$$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition:

Let U be open in $\mathbb{C}^n$, and suppose $F : U \rightarrow \mathbb{C}$ is smooth.
Then F is called **holomorphic** if, for all i,

$$\frac{\partial}{\partial \bar{z}_i} F = 0.$$

Complex manifolds

Definition:
A smooth manifold M admits a **complex structure** if there exists a covering by open subsets U_i with holomorphic coordinates.

Example (Complex projective line): $\mathbb{C}P^1 = (\mathbb{C}^2 \backslash \{(0,0)\})/\sim$, where $v_1 \sim v_2$ if, for some c in $\mathbb{C}^*$, $v_1 = cv_2$.

If $(z_0, z_1) \neq (0,0)$, denote its class by $[z_0 : z_1]$. Define

$$U_0 = \{[1, z]\}, \quad U_1 = \{[w, 1]\}$$

with coordinate charts $\phi_0([1, z]) = z, \quad \phi_1([w, 1]) = w.$

Then the transition function $(\phi_1 \circ \phi_0^{-1})(z) = 1/z$

is holomorphic where defined.

Generalized flag varieties

Let $K_{\mathbb{C}}$ be the complexification of K. So $Lie(K_{\mathbb{C}}) = \mathfrak{k}$, and $K \subset K_{\mathbb{C}}$.

With the choices of T and Δ^+ from before, we define the Borel subalgebra

$$\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{k}_{-\alpha}$$

with associated Borel subgroup $B = T_{\mathbb{C}} N$ in $K_{\mathbb{C}}$.

Definition:

The generalized flag variety $\mathcal{X}$ is the compact complex manifold $K_{\mathbb{C}}/B$.

Generalized flag varieties

Example: If $K = U(n)$ then $K_{\mathbb{C}} = GL(n, \mathbb{C})$.

- T= diagonal $(S^1)^n$,
- $\Delta = \{e_i - e_j \mid i \neq j\}$,
- $\Delta^+ = \{e_i - e_j \mid i < j\}$,
- Dominant weights $= \{k_1 e_1 + \cdots + k_n e_n \mid k_1 \geqslant \cdots \geqslant k_n\}$,
- B = lower triangular matrices,
- $\mathcal{X} = \{0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim(V_i) = i\}$.

Since $K \cap B = T$, $\quad K/T \cong K_{\mathbb{C}}/B$,
and we may pullback the complex structure to the left-hand side.

Example:. By Gram-Schmidt orthogonalization, $GL(n, \mathbb{C}) = U(n)B$. So

$$\mathcal{X} = U(n)/T = GL(n, \mathbb{C})/B.$$

Homogeneous holomorphic line bundles

Fix a character $(\chi_\lambda, \mathbb{C}_\lambda)$ of T with weight λ. $(\mathbb{C}_\lambda = \mathbb{C})$
We extend χ_λ to $T_\mathbb{C}$ holomorphically, and to N trivially.

Definition:

The homogeneous holomorphic line bundle $\mathcal{L}_\lambda$ on $K_\mathbb{C}/B$ is defined by

$$\mathcal{L}_\lambda = K_\mathbb{C} \times_B \mathbb{C}_\lambda = K_\mathbb{C} \times \mathbb{C}_\lambda / \sim,$$

where

$$(kb, z) \sim (k, \chi_\lambda(b)z)$$

$K_\mathbb{C}$ acts on the left by $k' \cdot (k, z) = (k'k, z)$.

Smooth sections of $\mathcal{L}_\lambda$

Suppose $F : K/T \to \mathcal{L}_\lambda$ is a smooth section defined by

$$F(k) = (k, s(k)).$$

Then

$$\begin{aligned} F(kt) = (kt, s(kt)) &= (k, s(k)) \\ \to \quad \chi_\lambda(t)s(kt) &= s(k) \\ \to \quad s(kt) &= \chi_\lambda(t)^{-1}s(k). \end{aligned}$$

Definition:

The **induced space** corresponding to smooth sections is given by

$$\mathrm{Ind}_T^K(\chi_\lambda) = \{s \in C^\infty(K) \mid s(kt) = \chi_\lambda(t)^{-1}s(k) \text{ for all } t \in T\}$$

with left action by K : $\quad (L(k')s)(k) = s(k'^{-1}k).$

Holomorphic sections

To test these functions for the holomorphic property, we recall that the space of anti-holomorphic tangent vectors is given by

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{k}_{-\alpha}.$$

That is, a holomorphic section $s(k)$ must satisfy

$$R(X_{-\alpha})\ s(k) = 0$$

for all α in Δ^+.

Definition:

Let $\Gamma(K/T, \mathcal{L}_\lambda)$ be the space of holomorphic sections of $\mathcal{L}_\lambda$.

The Borel-Weil theorem

Theorem (Borel-Weil):

With the definitions above, if λ is dominant then

$$\Gamma(K/T, \mathcal{L}_\lambda) = V_\sigma,$$

where (σ, V_σ) is an irreducible representation of K with highest weight λ. If λ is not dominant, it equals zero.

Intertwining operator for dominant λ:

$$\begin{array}{rcl} S : V_\sigma & \to & \Gamma(K/T, \mathcal{L}_\lambda) \\ v & \mapsto & Sv \end{array}$$

$$\boxed{(Sv)(k) = \langle \sigma(k)^{-1} v, \phi_\lambda \rangle.}$$

where ϕ_λ is a non-zero highest weight vector for V_σ.

Properties: $(Sv)(k) = \langle \sigma(k)^{-1} v, \phi_\lambda \rangle$

Intertwining property / K-equivariance:

$$\begin{aligned} [L(k')(Sv)](k) &= (Sv)(k'^{-1}k) \\ &= \langle \sigma(k)^{-1}\sigma(k')v, \phi_\lambda \rangle \\ &= (S[\sigma(k')v])(k). \end{aligned}$$

Section property:

$$\begin{aligned} (Sv)(kt) &= \langle \sigma(kt)^{-1}v, \phi_\lambda \rangle \\ &= \langle \sigma(k)^{-1}v, \sigma(t)\phi_\lambda \rangle \\ &= \langle \sigma(k)^{-1}v, \chi_\lambda(t)\phi_\lambda \rangle \\ &= \chi_\lambda(t)^{-1}(Sv)(k). \end{aligned}$$

Holomorphic property:

$$\begin{aligned} R(X_{-\alpha})(Sv)(k) &= -\langle \sigma(X_{-\alpha})\sigma(k)^{-1}v, \phi_\lambda \rangle \\ &= \langle \sigma(k)^{-1}v, \sigma(\bar{X}_{-\alpha})\phi_\lambda \rangle \\ &= -\langle \sigma(k)^{-1}v, \sigma(X_\alpha)\phi_\lambda \rangle \\ &= 0. \end{aligned}$$

Proof of vanishing and exhaustion

1) The Peter-Weyl Theorem implies that any section $S(k)$ in $\Gamma(K/T, \mathcal{L}_\lambda)$ is a sum of matrix coefficients.

2) Full reducibility under L implies that we can analyze the components in each $V_\pi \otimes (V_\pi)^*$ separately. Fix a nonzero highest weight vector ϕ_π for π under the L-action:

$$S_\pi(k) = \langle \pi(k)^{-1}\phi_\pi, v\rangle$$

3) For a nonzero holomorphic section, the above calculations imply that v must be a highest weight vector with weight λ.

If λ is dominant, we obtain all sections from the construction above.
If λ is not dominant, then no non-zero sections occur. QED

More complex geometry: smooth $(0, q)$-forms

Let M be a complex manifold.

In coordinates, we have the corresponding holomorphic and anti-holomorphic cotangent vectors

$$dz_i = dx_i + i dy_i, \quad d\bar{z}_i = dx_i - i dy_i,$$

and we extend the definition of smooth sections of a holomorphic line bundle to smooth differential forms of type $(0, q)$ with values in $\mathcal{L}$.

Locally, such a form looks like a sum of terms

$$s(p)\ d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q},$$

where $s(p)$ is a smooth section in coordinates.

Definition:

$A^{0,q}(M, \mathcal{L}) = \{\text{smooth forms on } M \text{ of type } (0, q) \text{ with values in } \mathcal{L}\}$

Dolbeault cohomology for holomorphic line bundles

Dolbeault cohomology = complex version of de Rham cohomology.

$\bar{\partial}$: analogue of d formed from anti-holomorphic tangent vectors.

- $\bar{\partial} : A^{0,q}(M,\mathcal{L}) \to A^{0,q+1}(M,\mathcal{L})$.
- $\bar{\partial}^2 = 0$
- $Z^{0,q}(M,\mathcal{L}) = Ker\{\bar{\partial} : A^{0,q}(M,\mathcal{L}) \to A^{0,q+1}(M,\mathcal{L})\}$
- $B^{0,q}(M,\mathcal{L}) = \bar{\partial}A^{0,q-1}(M,\mathcal{L}) \subset Z^{0,q}(M,\mathcal{L})$

Definition:

The $(0,q)$-th Dolbeault cohomology group on M with values in $\mathcal{L}$ is

$H^{0,q}(M,\mathcal{L}) = Z^{0,q}(M,\mathcal{L})/B^{0,q}(M,\mathcal{L})$.

Equivalent formulations for cohomology

Since K/T is compact, we have several equivalent cohomology theories:

- Sheaf cohomology $H^q(K/T, \mathcal{O}(\mathcal{L}_\lambda))$,
- Dolbeault cohomology $H^{0,q}(K/T, \mathcal{L}_\lambda)$,
- Harmonic forms $\mathcal{H}^{0,q}(K/T, \mathcal{L}_\lambda) = \text{Ker } \Box$.

In the last item,

$$\Box : A^{0,q}(K/T, \mathcal{L}_\lambda) \to A^{0,q}(K/T, \mathcal{L}_\lambda).$$

Hodge theory and harmonic forms

For harmonic forms,

- Since K/T has a K-invariant positive definite Hermitian metric, we can define a formal adjoint $\bar{\partial}^*$.
- The Laplace-Beltrami operator $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is elliptic.
- Solutions to $\square\omega = 0$ are smooth, and Ker $\square$ is finite dimensional.
- Ker $\square$ = Ker $\bar{\partial} \cap$ Ker $\bar{\partial}^*$.
- By sending each harmonic form to its cohomology class, we obtain an isomorphism

$$\mathcal{H}^{0,q}(K/T, \mathcal{L}_\lambda) \to H^{0,q}(K/T, \mathcal{L}_\lambda)$$

$$c \mapsto [c].$$

Anti-holomorphic cotangent vectors

Let $\{\omega_{-\alpha}\}$ be the basis for $\mathfrak{n}^*$ dual to the basis $\{X_{-\alpha}\}$ for $\mathfrak{n}$.

Action of T and $\mathfrak{b}$ on $\mathfrak{n}^*$:

$$Ad(t)\omega = \omega(Ad(t)^{-1}\ \cdot) \qquad ad(X)\omega = -\omega([X, \cdot])$$

- $\omega_{-\alpha}$ has weight α since

$$(ad(H)\omega_{-\alpha})(X_{-\alpha}) = -\omega_{-\alpha}([H, X_{-\alpha}]) = \alpha(H)$$

- $ad(X_{-\beta})\omega_{-\alpha}$ is zero or has weight $\alpha - \beta$.

  (If nonzero, $[X_{-\beta}, X_{-\alpha+\beta}]$ has weight $-\alpha$.)

Anti-holomorphic cotangent vectors

- Extend T-action to $\boxed{\wedge^q \mathfrak{n}^*}$: for instance,

$$Ad(t)(\omega_1 \wedge \omega_2 \wedge \omega_3) = Ad(t)\omega_1 \wedge Ad(t)\omega_2 \wedge Ad(t)\omega_3$$

- Leibniz rule for $ad(X)$ on $\wedge^q \mathfrak{n}^*$: for example,

$$\begin{aligned} ad(X)(\omega_1 \wedge \omega_2 \wedge \omega_3) \quad = \quad & ad(X)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ & + \omega_1 \wedge ad(X)\omega_2 \wedge \omega_3 \\ & + \omega_1 \wedge \omega_2 \wedge ad(X)\omega_3. \end{aligned}$$

- Interior product on $\wedge^q \mathfrak{n}^*$: Let $A \subseteq \Delta^+ \backslash \{\alpha\}$ and define

$$\omega_{-A} = \bigwedge_{\beta \in A} \omega_{-\beta} \in \wedge^q \mathfrak{n}^*.$$

Then define the interior product by

$$\begin{aligned} i(\omega_{-\alpha})(\omega_{-\alpha} \wedge \omega_{-A}) &= \omega_{-A} \\ i(\omega_{-\alpha})\omega_{-A} &= 0 \end{aligned}$$

Induced model for $A^{0,q}(K/T, \mathcal{L}_\lambda)$

For the induced model,

$A^{0,q}(K/T, \mathcal{L}_\lambda)$ corresponds to a subspace of $C^\infty(K) \otimes \wedge^q \mathfrak{n}^*$.
In particular, such functions $s(k)$ are smooth on K with values in $\wedge^q \mathfrak{n}^*$ such that

$$\boxed{s(kt) = ((\chi_\lambda \otimes Ad)(t^{-1}))\ s(k)}$$

with left action by

$$K : \quad (L(k')s)(k) = s(k'^{-1}k).$$

Weyl group action

Fix w in W. Use w also for its representative in $N_K(T)$.
Action of w on roots and weights:

$$(w\alpha)(H) = \alpha(w^{-1}Hw)$$

$$\boxed{\Delta(w^{-1}) = \{\alpha \in \Delta^+ \mid w\alpha < 0\ \}}$$

$$\boxed{\omega_{w^{-1}} = \bigwedge_{\substack{\alpha \in \Delta^+ \\ w\alpha < 0}} \omega_{-\alpha} \quad \in \wedge^{l(w)}\mathfrak{n}^*}$$

The weight of $\omega_{w^{-1}}$ is

$$\sum_{\substack{\alpha \in \Delta^+ \\ w\alpha < 0}} \alpha = \delta - w^{-1}\delta$$

where $\boxed{\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.}$ (Knapp reference, Prop. 3.19, p. 137)

The Borel-Weil-Bott theorem

For w in W, define $\boxed{l(w) = \#\{\alpha \in \Delta^+ \mid w\alpha < 0\}}$.

Theorem (Borel-Weil-Bott):

With the above definitions and choices, we have

- If $\langle \lambda + \delta, \alpha \rangle = 0$ for some α in Δ, then $H^{0,k}(K/T, \mathcal{L}_\lambda) = 0$ for all k.
- Otherwise, there exists a unique w in W such that $\mu = w(\lambda + \delta) - \delta$ is dominant with respect to Δ^+, and

$$H^{0,k}(K/T, \mathcal{L}_\lambda) = \begin{cases} 0, & k \neq l(w), \\ V_\sigma, & k = l(w), \end{cases}$$

  where (σ, V_σ) is the irreducible representation with highest weight μ.

Explicit harmonic forms

Intertwining operator: (Griffith-Schmid (1969), D. (2000))

Let ϕ_μ be a non-zero highest weight vector for σ.

$$\begin{array}{rcl} S : V_\mu & \to & \mathcal{H}^{0,l(w)}(K/T, \mathcal{L}_\lambda) \\ v & \mapsto & Sv \end{array}$$

$$\boxed{(Sv)(k) = \langle \sigma(k)^{-1} v, \sigma(w)^{-1}\phi_\mu \rangle \otimes \omega_{w^{-1}}}$$

Section property:

$$\begin{aligned} (Sv)(kt) &= \langle \sigma(kt)^{-1} v, \sigma(w)^{-1}\phi_\mu \rangle \otimes Ad(t)^{-1} Ad(t)\omega_{w^{-1}} \\ &= \langle \sigma(k)^{-1} v, \sigma(t)\sigma(w)^{-1}\phi_\mu \rangle \otimes Ad(t)^{-1} \chi_{\delta - w^{-1}\delta}(t)\omega_{w^{-1}} \\ &= \langle \sigma(k)^{-1} v, \sigma(w)^{-1}\sigma(wtw^{-1})\phi_\mu \rangle \otimes Ad(t)^{-1} \chi_{\delta - w^{-1}\delta}(t)\omega_{w^{-1}} \\ &= (\chi_{w^{-1}\mu + w^{-1}\delta - \delta} \otimes Ad)(t^{-1})(Sv)(k) \end{aligned}$$

Harmonic Property: $\bar{\partial} S v = 0$

Formula in our setting:
Griffiths-Schmid (1969), p. 260, (1.6b), and p. 278, (5.1) and (5.2).

$$\bar{\partial}\, f\, \omega \quad = \quad \sum_{\alpha \in \Delta^+} X_{-\alpha} f \;\; \omega_{-\alpha} \wedge \omega \quad + \quad \frac{1}{2} \sum_{\alpha \in \Delta^+} f\; \omega_{-\alpha} \wedge ad(X_{-\alpha})\omega$$

Key observation:

$\bar{\partial} S v = 0$ without cancellation

that is, all terms in both sums equal 0.

First sum, part 1: $\omega_{-\alpha} \wedge \omega_{w^{-1}}$

- If α is in $\Delta(w^{-1})$ then $w\alpha < 0$ and $\omega_{-\alpha} \wedge \omega_{w^{-1}} = 0$ by definition.

Harmonic Property: $\bar{\partial} S v = 0$

First sum, part 2: $X_{-\alpha}\langle \sigma(k)^{-1}v, \sigma(w)^{-1}\phi_\mu\rangle$

Same calculation as in the Borel-Weil theorem.

- If α is not in $\Delta(w^{-1})$, then $w\alpha > 0$ and

$$\begin{aligned} X_{-\alpha}\langle \sigma(k)^{-1}v, \sigma(w)^{-1}\phi_\mu\rangle &= -\langle \sigma(X_{-\alpha})\sigma(k)^{-1}v, \sigma(w)^{-1}\phi_\mu\rangle \\ &= -\langle \sigma(k)^{-1}v, \sigma(X_\alpha)\sigma(w)^{-1}\phi_\mu\rangle \\ &= -\langle \sigma(k)^{-1}v, \sigma(w)^{-1}\sigma(cX_{w\alpha})\phi_\mu\rangle \\ &= 0 \end{aligned}$$

since $w\alpha > 0$ and ϕ_μ is a highest weight vector.

Harmonic Property: $\bar{\partial} S v = 0$

$$\frac{1}{2} \sum_{\alpha \in \Delta^+} f \, \omega_{-\alpha} \wedge ad(X_{-\alpha})\omega$$

Second sum, part 1: If $w\alpha < 0$ then $\omega_{-\alpha} \wedge \omega_{w^{-1}} = 0$, same as before.

Second sum, part 2: If $w\alpha > 0$ then $ad(X_{-\alpha})\omega_{w^{-1}} = 0$

- Leibniz rule for $ad(X_{-\alpha})$
- If nonzero, $ad(X_{-\alpha})\omega_{-\beta}$ is a multiple of $\omega_{-\beta+\alpha}$.
- If $w\beta < 0$ then $w(\beta - \alpha) < 0$ and $\omega_{-\beta+\alpha}$ is distinct from $\omega_{-\beta}$.

  Thus the wedge product vanishes by definition.

Harmonic Property: $\bar{\partial}^* Sv = 0$

Formula in our setting: Griffiths-Schmid, p. 279, (5.8).

$$\boxed{\bar{\partial}^* \; f \; \omega \quad = \quad \sum_{\alpha \in \Delta^+} -X_\alpha f \quad i(\omega_{-\alpha})\omega \quad + \quad \frac{1}{2} \sum_{\alpha \in \Delta^+} \; f \; i(\omega_{-\alpha}) ad(X_\alpha)\omega}$$

Key observation:

$\boxed{\bar{\partial}^* Sv = 0 \text{ without cancellation}}$
that is, all terms in each sum equal 0.

First sum, part 1: $\quad i(\omega_{-\alpha})\omega_{w^{-1}}$

- If α is not $\Delta(w^{-1})$ then $w\alpha > 0$ and $i(\omega_{-\alpha})\omega_{w^{-1}} = 0$ by definition.

Harmonic Property: $\bar{\partial}^* Sv = 0$

First sum, part 2: $-X_\alpha \langle \sigma(k)^{-1} v, \sigma(w)^{-1} \phi_\mu \rangle$

- If α is in $\Delta(w^{-1})$, then $w\alpha < 0$ and

$$\begin{aligned} -X_\alpha \langle \sigma(k)^{-1} v, \sigma(w)^{-1} \phi_\mu \rangle &= \langle \sigma(X_\alpha) \sigma(k)^{-1} v, \sigma(w)^{-1} \phi_\mu \rangle \\ &= \langle \sigma(k)^{-1} v, \sigma(X_{-\alpha}) \sigma(w)^{-1} \phi_\mu \rangle \\ &= \langle \sigma(k)^{-1} v, \sigma(w)^{-1} \sigma(cX_{-w\alpha}) \phi_\mu \rangle \\ &= 0 \end{aligned}$$

since $-w\alpha > 0$ and ϕ_μ is a highest weight vector.

Harmonic Property: $\bar{\partial}^* Sv = 0$

$$\frac{1}{2}\sum_{\alpha\in\Delta^+} f \; i(\omega_{-\alpha})ad(X_\alpha)\omega$$

Second sum, part 1: If $w\alpha > 0$ then $i(\omega_\alpha)\omega_{w^{-1}} \; = \; 0$ by definition.

Second sum, part 2: If $w\alpha < 0$ then $ad(X_\alpha)\omega_{w^{-1}} \; = \; 0$.

- First, Leibniz rule for $ad(X_\alpha)$.
- If nonzero, $ad(X_\alpha)\omega_{-\beta}$ is a multiple of $\omega_{-\beta-\alpha}$.
- If $w\beta < 0$ then $w(\beta + \alpha) < 0$ and $\omega_{-\beta-\alpha}$ is distinct from $\omega_{-\beta}$.

Thus the wedge product vanishes by definition. QED

$\mathfrak{n}$-cohomology and Kostant's Theorem

$$L^2(K) \otimes \wedge^q \mathfrak{n}^* \quad \cong \quad \bigoplus_{\sigma \in \hat{K}} V_\sigma \otimes \boxed{(V_\sigma)^* \otimes \wedge^q \mathfrak{n}^*}$$

All computations for geometry are actions on the boxed factor.
For our form, this gives the Kostant class

$$\langle \,\cdot\, , \sigma(w)^{-1} \phi_\mu \rangle \otimes \omega_{w^{-1}},$$

as given in Kostant (1961), Theorem 5.14.

$\mathfrak{n}$-cohomology and Kostant's Theorem

$$\langle \,\cdot\, , \sigma(w)^{-1}\phi_\mu \rangle \otimes \omega_{w^{-1}}$$

If we identify $\wedge^q \mathfrak{n}^*$ with $(\wedge^q \mathfrak{n})^*$, this becomes an element of

$$Hom_{\mathbb{C}}(\wedge^q \mathfrak{n}, (V_\sigma)^*).$$

This space has corresponding actions of T and $\mathfrak{b}$ and an analogue of $\bar{\partial}$.

In turn, these *Hom* spaces are the cochain spaces for $\mathfrak{n}$-cohomology.

Kostant's theorem gives a calculation of these cohomology groups, the statement of which mirrors the Borel-Weil-Bott theorem.

References and Further Reading

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Thank you!

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Comments and Questions: Added 5/13/2025

- See S. Kumar's lecture notes (Borel-Weil-Bott theorem and geometry of Schubert varieties, 2012, available online) for an algebro-geometric exposition on the theorem. See also Hirzebruch's Toplogical Methods in Algebraic Geometry for background on Riemann-Roch theorems.

- $l(w)$ also equals the number of simple reflections in any minimal product expansion of w into simple reflections.

- What does higher cohomology measure in the BWB Theorem/ Kostant's Theorem?

  We obtain holomorphic sections when the positive system Δ^+ and λ are perfectly aligned: each X_α annihilates ϕ_λ. As seen in the calculations for higher cohomology, the extreme vector $\sigma(w)^{-1}\phi_\mu$ only vanishes under some raising operators X_α, and the defect is measured precisely by the roots that contribute to $\omega_{w^{-1}}$.

Comments and Questions: Added 5/13/2025

- Induced representations: Consider $K_{\mathbb{C}}/B = \mathbb{C}P^1$, and let $p(z_1, z_2)$ be a homogenous polynomial of degree N in z_1 and z_2. Then

$$p(cz_1, cz_2) = c^N p(z_1, z_2).$$

This polynomial pulls back to a function on $K_{\mathbb{C}} = GL(2, \mathbb{C})$ by $P\begin{bmatrix} * & z_1 \\ * & z_2 \end{bmatrix} = p(z_1, z_2)$. Then

$$P(\begin{bmatrix} * & z_1 \\ * & z_2 \end{bmatrix}\begin{bmatrix} a & 0 \\ * & c \end{bmatrix}) = P\begin{bmatrix} * & cz_1 \\ * & cz_2 \end{bmatrix} = c^N P\begin{bmatrix} * & z_1 \\ * & z_2 \end{bmatrix}$$

corresponds to $P(gb) = \chi(b)^{-1} P(g)$ with $\chi(b) = c^{-N}$ and $\lambda = -Ne_2$.
This is the weight of the highest weight vector z_2^N.
All other weights are of the form $\lambda - k(e_1 - e_2)$.

Comments and Questions: Added 5/13/2025

- Intuition for harmonic forms: a difficulty with cohomology classes is knowing when a cocycle is a coboundary. Hodge theory solves this issue by finding an orthocomplement to the space of coboundaries in the space of cocycles. In general, harmonic forms are not canonical and depend on the choice of metric.

- Sheaf theory: to convert between cochains in Čech and de Rham cohomology (and likewise Dolbeault cohomology), see, for instance, Appendix 1, 3.4, p. 224 of Kirillov, Lectures on the Orbit Method, 2004 (book).

- The BWB theorem extends to G/P for both line bundles and vector bundles with irreducible finite-dimensional fiber for L when $P = LU$. See D. (2000) for the corresponding adjustments to the explicit formula in the latter case.