A visual approach to symmetric chain decompositions of finite Young lattices

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Credits

Queens Experience in Discrete Mathematics REU (QED) (York College): NSF-DMS 2150251

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Terrence Coggins (York College, Spring 2024)
Ammara Gondal (BMCC, Spring/Summer 2024)
Arnav Krishna (Harvard, Summer 2024)

Preprint: https://arxiv.org/abs/2407.20008

Girls' Angle Bulletin

5 part series on integer partitions for math circles with K-12 students (First part: June / July 2023)

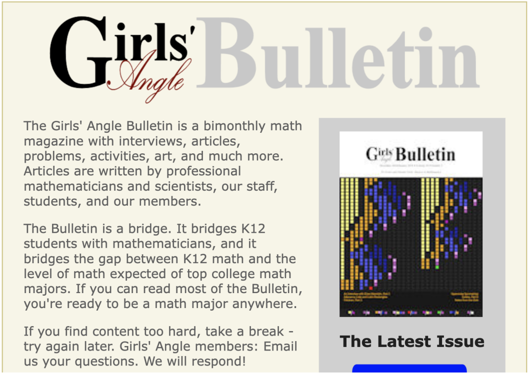


Figure: https://www.girlsangle.org/page/bulletin.php

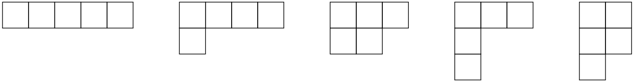
Integer partitions and Young diagrams

Definition: Let k be a non-negative integer.
An **(integer) partition** λ of k into m **parts** is a set of m positive integers λ_i such that

$$\lambda_1 + \cdots + \lambda_m = k.$$

Example (Young diagrams):
The partitions of 5 with at most 3 parts are

5   41   32   311   221.



Partial order on the finite Young lattice $L(m, n)$

Definition:. Let $L(m, n)$ be the set of partitions with at most m parts, each of size at most n. That is, the Young diagrams for these partitions fit into an $m \times n$ rectangle.

It will be convenient to add zeros when partitions have fewer than m parts, and to define the empty partition $\varnothing$ of 0 with zero parts.

$L(m, n)$ admits a **partial order** using entry-wise comparison:

$$\lambda \leqslant \mu \quad \text{if each} \quad \lambda_i \leqslant \mu_i.$$

For the corresponding Young diagrams, λ fits inside of μ.

Finite Young lattices $L(m, 2)$

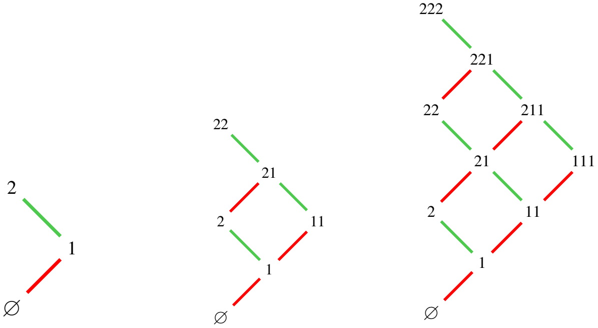


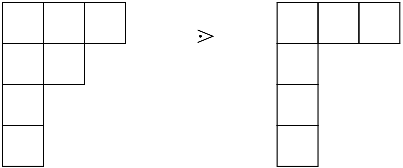
Figure: Finite Young lattices $L(1, 2)$, $L(2, 2)$, $L(3, 2)$

Edge colors: from top to bottom,
Green - replace 2 with a 1
Red - remove a 1

Covering Relations

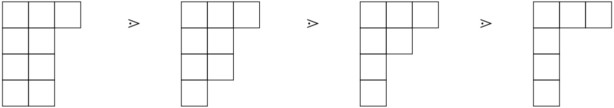
Definition: We say μ **covers** λ ($\mu \gtrdot \lambda$) if λ is obtained from μ by reducing one part of μ by 1.
For the corresponding Young diagrams, we remove one square from μ.

$$3211 \gtrdot 3111$$



Saturated Chains

Definition:. A **saturated chain** in $L(m, n)$ is a sequence of consecutive coverings.



$$3222 \gtrdot 3221 \gtrdot 3211 \gtrdot 3111$$

Third way: List multiplicity of each part by decreasing size
$\longrightarrow$ weak compositions. (more later)

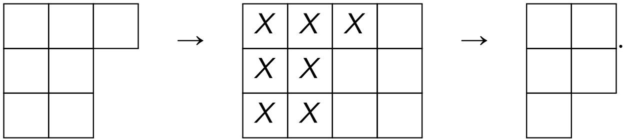
$$1300 \gtrdot 1210 \gtrdot 1120 \gtrdot 1030$$

Symmetric chain decomposition

Definition:. A **symmetric chain decomposition** on $L(m, n)$ is

- a collection of saturated chains such that
- each vertex used exactly once and
- each chain begins and ends on opposite levels

Note that $L(m, n)$ is dual to itself (vertically symmetric) under complementation: in $L(3, 4)$,



Finite Young lattices $L(m, 2)$

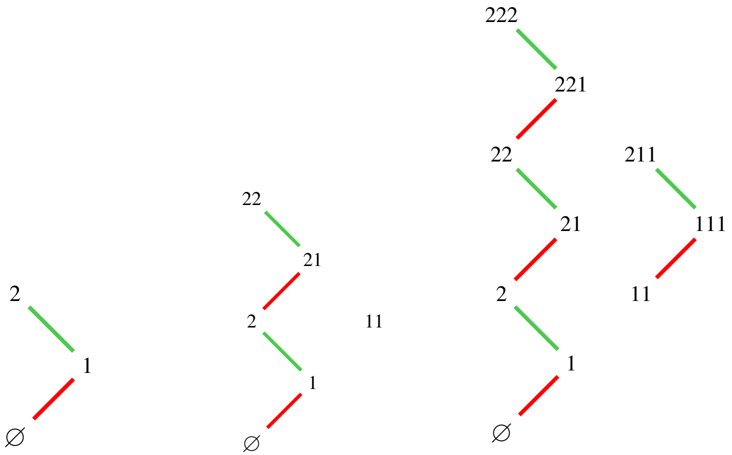


Figure: Symmetric chain decompositions for $L(1,2)$, $L(2,2)$, $L(3,2)$

Stanley's Conjecture for $L(m, n)$ (1980)

R. Stanley (1980):
$L(m, n)$ admits a symmetric chain decomposition for all m, n.

Known cases:

- $n = 3$: Lindström (1980)
- $n = 4$: West (1980)
- $n = 5$: Wen (2024)

Also recent progress:

Orellana, R., Saliola, F., Schilling, A., and Zabrocki, M.

From quasi-symmetric to Schur expansions with applications to symmetric chain decompositions and plethysm, arXiv:2404.04512v1, 27pp, 2024.

Weak compositions

Definition:. A **weak composition** of m with $n+1$ parts is an ordered sum of non-negative integers

$$x_1 + x_2 + \cdots + x_{n+1} = m$$

Note that 0 is allowed as a part.

Convert elements of $L(m,n)$ to weak compositions by recording the multiplicities of parts in reverse order

In L(6, 3): 332220 → 2301

(Two 3s, 3 2s, no 1s, and one 0.)
332220 is a partition of 12 with 6 parts.
The sum of the parts in the weak composition 2301 is 6.

Example: $L(3,3)$

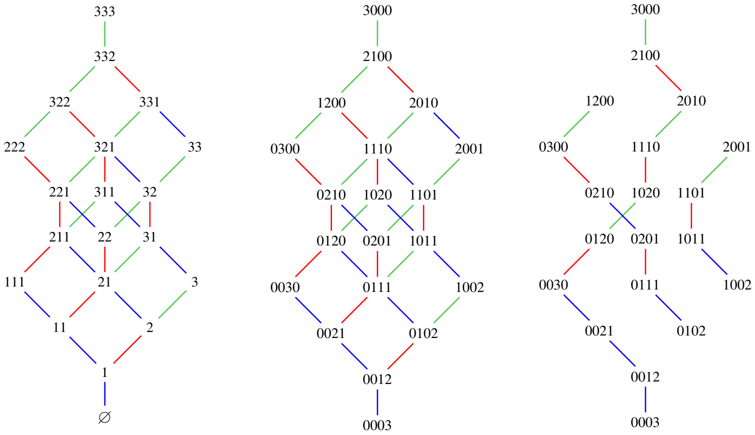


Figure: Finite Young lattice $L(3,3)$ with colored links

Coloring of edges

For $L(3,3)$, three colors for edges based on covering (partition / Young diagram / weak composition)

- Green: remove a 3 / remove square from far right / shift a 1 from 1st to 2nd position,
- Red: remove a 2 / remove square from middle / shift a 1 from 2nd to 3rd position,
- Blue: remove a 1 / remove square from far left / shift a 1 from 3rd to 4th position

Weak compositions for $L(3,3)$: ordering on A_3 weight diagrams using simple roots $e_1 - e_2, e_2 - e_3, e_3 - e_4$

Example: Weight string through highest weight $\beta = 3e_1$ along $e_1 - e_2$

$$(3,0,0,0) \gtrdot (2,1,0,0) \gtrdot (1,2,0,0) \gtrdot (0,3,0,0)$$

Main Results: (QED, 2024)

1) As weak compositions, $L(m,n)$ takes the form of an n-simplex dilated by a factor of m.

2) Alternatively, the ordering is a weight diagram for type A_n with highest weight me_1.

Recall that the root system of type of type A has

1. Roots. $\Delta = \{e_i - e_j\}$.
2. .Simple roots. $\Pi = \{e_i - e_{i+1}\}$, and
3. Fundamental weights $\{e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots\}$

3) New pictures

Example: $L(3,3)$

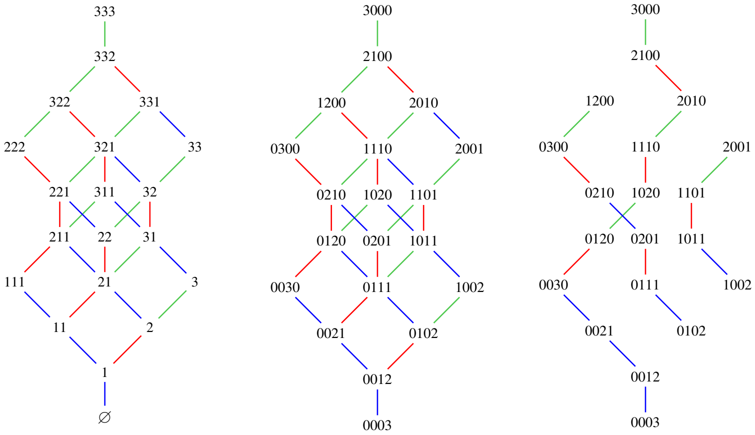


Figure: Finite Young lattice $L(3,3)$ with colored links

Weight strings along simple roots

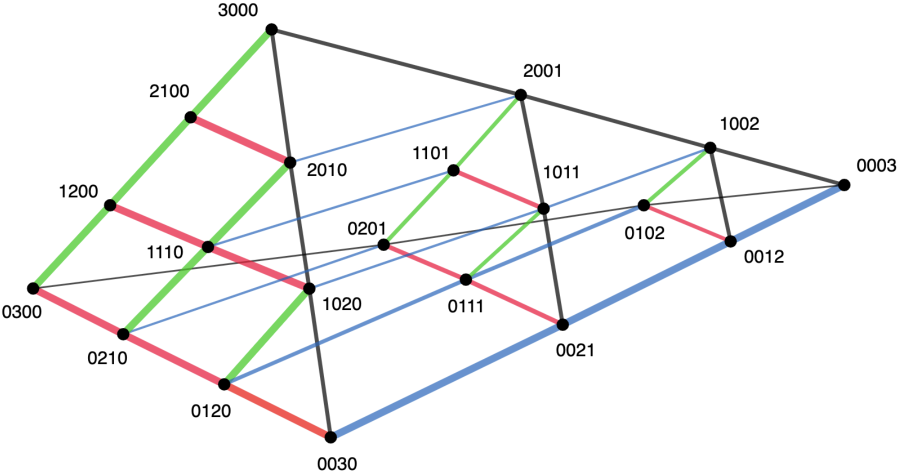


Figure: $L(3,3)$ with weight strings

Lindström's Algorithm for $L(m, 3)$

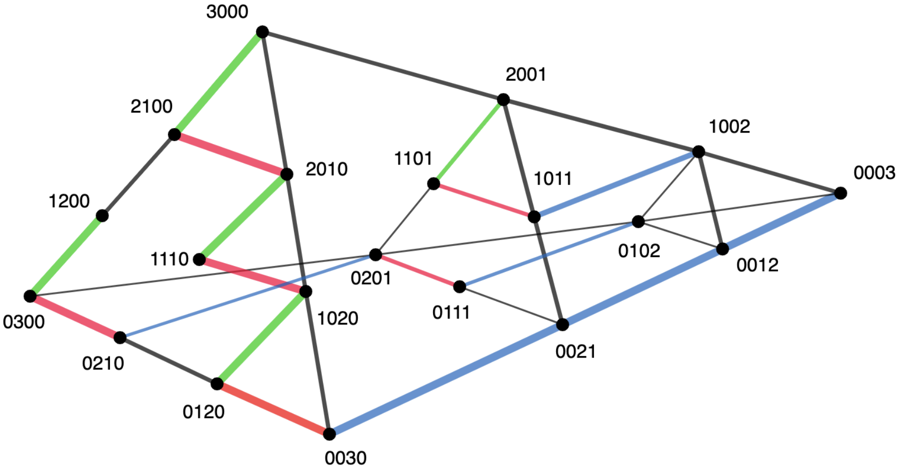


Figure: Symmetric chain decomposition for $L(3, 3)$

Lindström's Algorithm for $L(m, 3)$

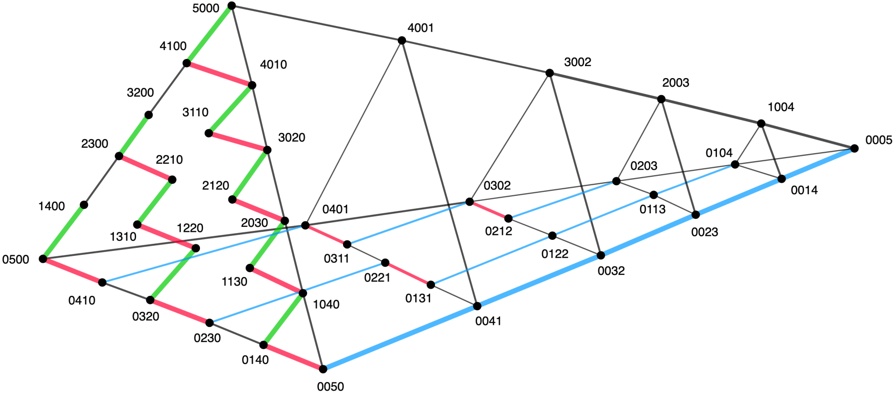


Figure: Partial symmetric chain decomposition for $L(5, 3)$

Thank you!

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