Lattice path enumeration for semi-magic squares by Latin rectangles

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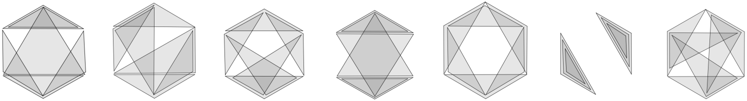


Prelude:

Question: Up to equivalence, how many ways can one put six triangles inside a regular hexagon such that three triangles abut each vertex?

Answer: Seven (or six)

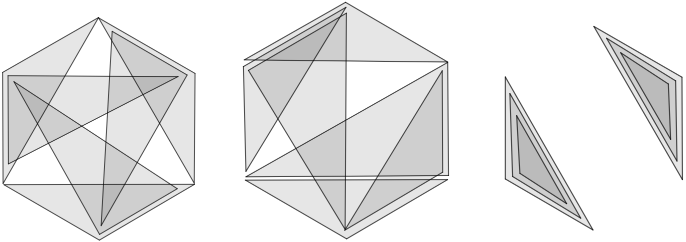


Table of Contents:

Joint work with Won Geun Kim

arXiv:
1. Donley and Kim (February 2022),
Lattice path enumeration for semi-magic squares by Latin rectangles.

2. Donley (July 2021), LPE for SMS of size three

Main theme of Talk

Interconnection between:

- semi-magic squares with 0/1 entries,
- Latin rectangles and Latin squares, and
- k-uniform, k-regular hypergraphs.
- (Bipartite/bicolored graphs, ear diagrams)

- Recreational/"Actuarial"
- Program: $M(3) \quad \rightarrow \quad$ Clebsch-Gordan coefficients for $SU(2)$

Semi-magic squares of size n

A square matrix M is called a **semi-magic square** if

- entries are integers $\geqslant 0$, and
- the sum along any row or column is equal to the same number L.

$L = \rho(M)$ is called the **line sum** of M.

Let $M(n)$ be the monoid (addition) of all semi-magic squares of size n.

$$M = \begin{bmatrix} 3 & 2 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 4 \end{bmatrix}, \qquad L = 9$$

Examples: $L = 1$

A **permutation matrix** is a square matrix such that there is exactly one 1 in each row and column.

$$P_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{(132)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$P_{(13)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{(23)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Algebra for Semi-Magic Squares

1) If $k \geqslant 0$ and M, N are semi-magic squares, so are

$$kM, \qquad M + N$$

with line sums kL_M and $L_M + L_N$, respectively.

2) Any linear combination of permutation matrices with $a_i \geqslant 0$ integers

$$a_1 P_{\sigma_1} + \cdots + a_t P_{\sigma_t}$$

is a semi-magic square with line sum $a_1 + \cdots + a_t$.

3) **Birkhoff-von Neumann:**
Every semi-magic square of size n may be written in this form.

Linearly Independent? No.

Example: $n = 3$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}+\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}+\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$= \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = J$$

For $n \geqslant 4$,

$$n! \; > \; n^2 \; > \; 1 + (n-1)^2.$$

In general: R. Stanley → Methods of commutative algebra (syzygies)
Proof of the Anand-Dumir-Gupta Conjecture

Graded Poset $M(n)$

$M(n)$ forms a graded poset:
Partial ordering (entry-wise for all entries):

$$M \leqslant N \quad \text{if} \quad m_{ij} \leqslant n_{ij}$$

$$M \lessdot N \quad \text{if} \quad N = M + P_\sigma \text{ for some } \sigma \in S_n$$

Rank function $\rho(M) = L$

Sequence of nested self-dual downsets in $M(n)$:
$M(n, s)$: semi-magic squares of size n with maximum entry $\leqslant s$
or $M \leqslant sJ$

Finite graded poset with $\hat{0}$ and $\hat{1}$

A useful analogy for $s = 1$

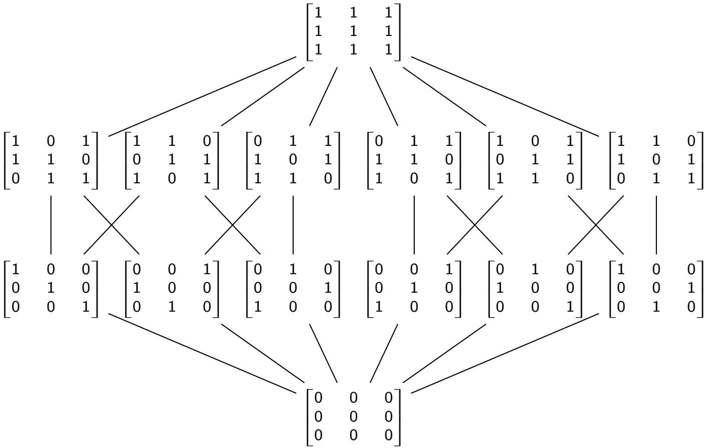
$(a_1, a_2, \ldots, a_n)$:

Boolean poset $(a_i = 0, 1)$ $\subset$ weak compositions $(a_i \geqslant 0)$

M :

(0, 1)-semi-magic squares $\subset$ semi-magic squares

Poset diagram for $M(3,1)$



Maximal chains/lattice paths?

Another useful analogy: Paths in the Young Lattice

Young lattice: graded by number of boxes in Young diagram

Paths in the Young lattice are enumerated by standard Young tableaux. (Young diagrams with fillings strictly increasing in rows and columns)

$$\hat{0} \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}.$$

The number of standard Young tableaux of a given shape are counted by the hook length formula.

Paths in $M(3,1)$

By Birkhoff-von Neumann, semi-magic squares are constructed by iterated addition of permutation matrices.

If we use single line notation for permutations,
the maximum one property forces the Latin rectangle property on the list.

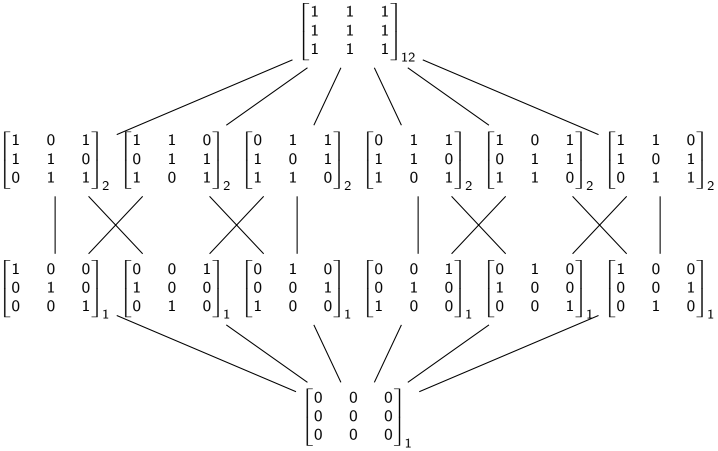
$$\hat{0} \longrightarrow \begin{matrix} 2 & 3 & 1 \end{matrix} \longrightarrow \begin{matrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{matrix} \longrightarrow \begin{matrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Latin squares represent distinct paths from $\hat{0}$ to J in $M(n,1)$

OEIS (A002860): 1, 2, 12, 576, 161280, 812851200, ... (11 entries)

Counting paths in $M(3,1)$



Order-raising operator (Pascal's triangle rule)

$n \geqslant 4$: pass to orbits for large group

Wreath Product $G = S_n \wr \mathbb{Z}/2$

At matrix level, the magic square property and line sum are preserved by

1. row permutations,
2. column permutations, and
3. transpose.

$|G| = 2(n!)^2$
Faithful action on $M(n,1)$: σ, τ in S_n

$$R(\sigma)C(\tau) \cdot M = P_\sigma M P_\tau^{-1}$$

$$R(\sigma)C(\tau)T \cdot M = P_\sigma M^T P_\tau^{-1}$$

$$R(\sigma)C(\tau) = C(\tau)R(\sigma), \qquad R(\sigma)T = TC(\sigma)$$

Orbits under G

Homogeneity will drastically reduce the amount of data to track (preserves $\leqslant$, $\rho(M)$, same downset structure, same path numbers)

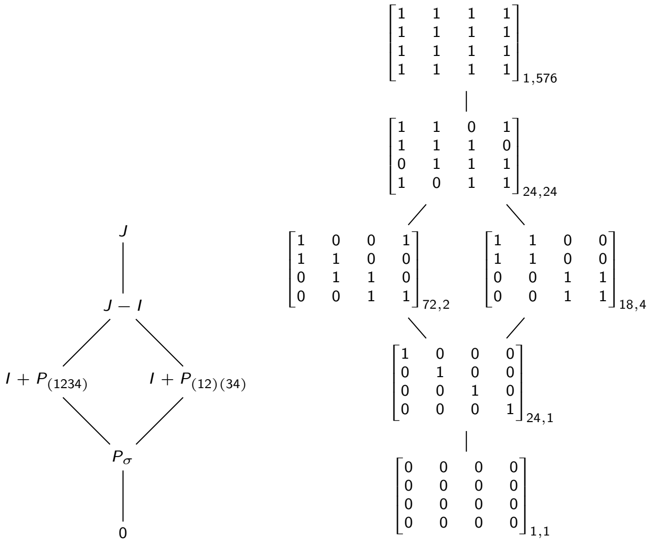
To count paths in $M(n, 1)$ using orbits, we need

- good representatives for orbits,
- orbit size o_M,
- covering data,
- path numbers $v(M)$ of elements directly below.

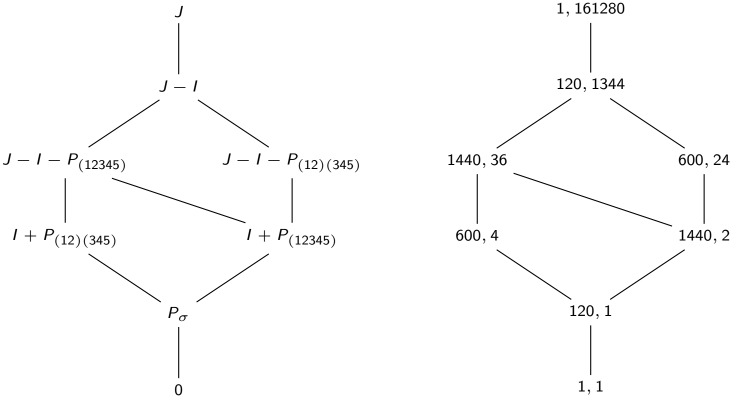
Orbit-Stabilizer Theorem

$$o_M = \frac{|G|}{|Stab_G(M)|}$$

Main Results: Poset diagram of orbits for $M(4,1)$

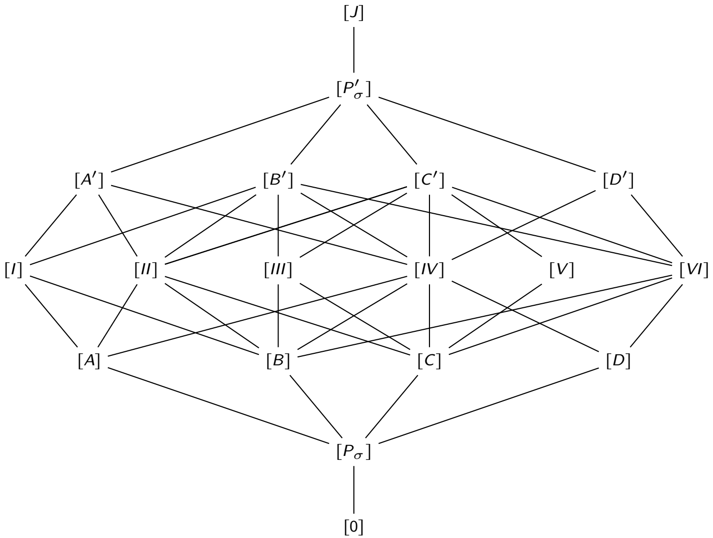


Main Results: Poset diagram of orbits for $M(5,1)$



At right: (size of orbit o_M, path number $v(M)$)

Main Results: Poset diagram of orbits of $M(6,1)$



Main Results: Orbit data for $M(6,1)$

M	$\rho(M)$	o_M	$v(M)$	M	$\rho(M)$	o_M	$v(M)$
$\hat{0}$	0	1	1	P_σ	1	720	1
A	2	16200	4	I	3	86400	48
B	2	43200	2	II	3	129600	48
C	2	7200	4	III	3	16200	48
D	2	1350	8	IV	3	43200	72
A'	4	16200	4224	V	3	200	144
B'	4	43200	4032	VI	3	21600	48
C'	4	7200	4608	P'_σ	5	720	1128960
D'	4	1350	5376	J	6	1	812851200

$A:\ (12)(3456), \quad B:\ (123456), \quad C:\ (123)(456), \quad D:\ (12)(34)(56)$

$M(4,1)$: Ranks 0, 1, $n-1$, n

Poset diagram of $M(n,1)$
Self-dual using the involution:

$$M \to M' = J - M, \qquad \rho(M') = n - \rho(M)$$

- Rank 0: 1 element, 1 orbit, 1 path
- Rank 1: $n!$ elements, 1 orbit, 1 path

- Rank $n-1$: $n!$ elements, 1 orbit, paths: use rank $n-2$,
- Rank n: 1 element, 1 orbit, paths: $n!$ times $v(J-I)$

$M(4,1)$: Rank 2

Orbits in rank 2: represented by a (unique) derangement class. Recall that a derangement has no fixed points.

$$P_\tau + P_\sigma = R(\tau) \cdot (I + P_{\tau^{-1}\sigma}).$$

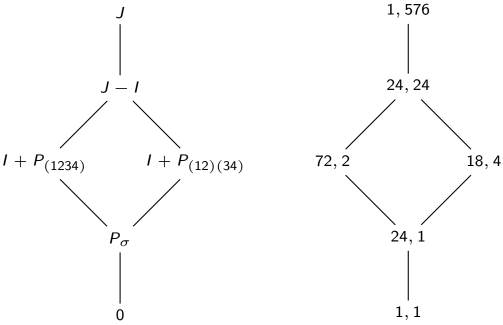
Orbit size: Stabilizers of $M = I + P_\sigma$ are elementary in our cases.

Path numbers:
Switching rule: if $\sigma = \sigma'\sigma''$ as a disjoint product of cycles, then

$$I + P_\sigma = P_{\sigma'} + P_{\sigma''}.$$

That is, if σ may be written as a product of c disjoint cycles, there are 2^c paths from $\hat{0}$ to $I + P_\sigma$

Poset diagram of $M(4, 1)$



$M(4,1)$: Covering Data for Rank 3

Homogeneity → only need covering data for orbit representatives

- set up tree to determine which permutation matrices can be removed
- given such a permutation matrix, determine remaining class when extracted

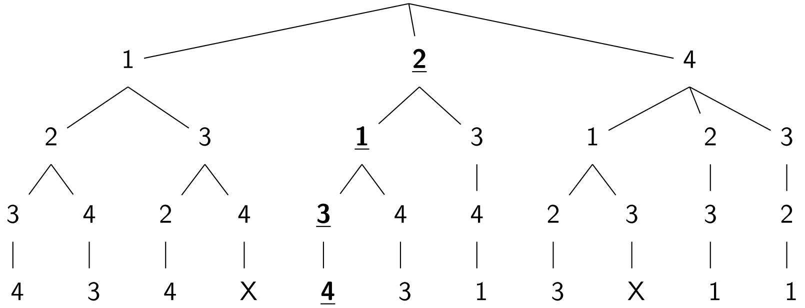
Remove 2134 from $J - P_{(13)(24)}$:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{matrix} 1 & 2 & \boxed{3} & \boxed{4} \\ \boxed{2} & 3 & 4 & 1 \\ 4 & \boxed{1} & 2 & 3 \end{matrix} \longrightarrow \begin{matrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{matrix}$$

Final answer:

$$M = P_{(34)} + P_{(14)(23)} = C(34) \cdot (I + P_{(1423)})$$

$M(4,1)$: Tree for downward steps at Rank 3



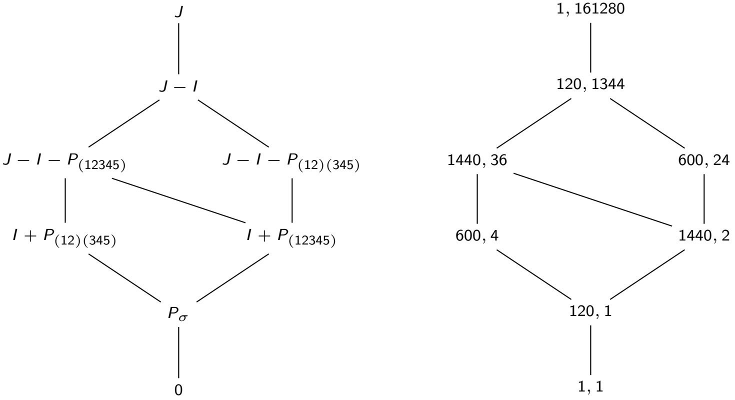
1234, 1243, 1324, [2134], 2143,
2341, 4123, 4231, 4321

$\rightarrow$

1	2	[3]	[4]
[2]	3	4	1
4	[1]	2	3

$$v(J - P_{(13)(24)}) = 6 \cdot 2 + 3 \cdot 4 = 24$$

Poset diagram of orbits for $M(5,1)$



At right: (size of orbit o_M, path number $v(M)$)

Poset diagram of orbits of $M(6,1)$

$[J]_{1,812851200}$

$[P'_\sigma]_{720,1128960}$

$[A']_{16200,4224}$ $[B']_{43200,4032}$ $[C']_{7200,4608}$ $[D']_{1350,5376}$

$[I]_{86400,48}$ $[II]_{129600,48}$ $[III]_{16200,48}$ $[IV]_{43200,72}$ $[V]_{200,144}$ $[VI]_{21600,48}$

$[A]_{16200,4}$ $[B]_{43200,2}$ $[C]_{7200,8}$ $[D]_{1350,8}$

$[P_\sigma]_{720,1}$

$[0]_{1,1}$

$M(6,1)$: Rank 3

Same general idea as before, but rank 3 needs new ideas:

- representatives for orbits,
- stabilizers for orbit size, and
- path numbers in ranks 3 and 4.

Key idea: treat M as incidence matrix for **hypergraph** H

Semi-magic property: H is both k-**uniform** and k-**regular.**

$n = 6$: hexagon
3-uniform: hyperedges are triangles,
3-regular: 3 triangles at each vertex.

7 hypergraphs $\to$ 6 orbits (transpose of M merges two classes)

$M(6,1)$: Group action on hypergraphs

Technically, hypergraph pair (H, H^*): M and M^T

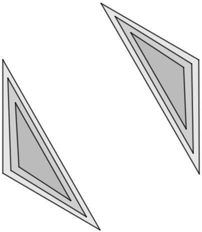
Group acts on hypergraph pair through M:

- $R(\sigma)$: changes vertex labels in H, hyperedge labels in H^*,
- $C(\tau)$: change vertex labels in H^*, hyperedge labels in H,
- T: exchange H and H^*

For rank 3 in $M(6,1)$, all but one orbit are self-dual under transpose.

V

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{array} \to$$



- Distinguishing feature: a pair of trebly nested triangles
- Stabilizer: $(S_3)^4$, normalized by exchange $\to$ 5184 elements

Transpose in same class: twice stabilizer of hypergraph

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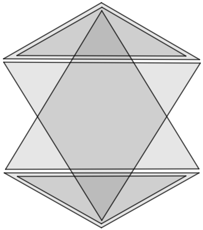
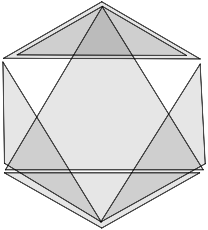
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 3 & 4 \\ 6 & 1 & 5 & 3 & 4 & 2 \end{matrix} \rightarrow$$



- Distinguishing feature: a pair of doubly nested triangles, two triple edges
- Stabilizer: $(\mathbb{Z}/2)^4$, normalized by exchange $\rightarrow$ 64 elements

Ia

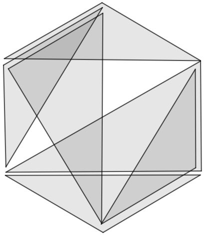
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 3 & 1 & 5 \\ 6 & 1 & 5 & 2 & 3 & 4 \end{matrix} \rightarrow$$



- Distinguishing feature:
  pair of doubly nested triangles, three-cycle of shared edges
- Stabilizer: $S_3 \times \mathbb{Z}/2 \rightarrow 12$ elements
- Transpose not in same class

Ib

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 6 & 1 \\ 6 & 3 & 2 & 5 & 1 & 4 \end{matrix} \to$$

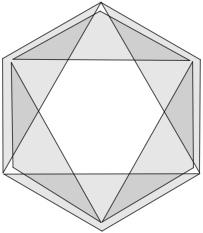


- Transpose in same class as *Ia*
- Distinguishing feature:

  one triple edge, three double edges meet at vertex
- Stabilizer: same as *Ia* $\to$ 12 elements

IV

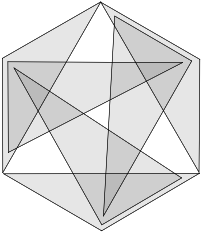
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{matrix} \to$$



- Distinguishing feature: six-cycle of shared edges
- Stabilizer: $D_{12} \to 24$ elements

VI

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{matrix} \rightarrow$$



- Distinguishing feature: three disjoint double edges
- Stabilizer: $(\mathbb{Z}/2)^3$, normalized by S_3 (almost) $\rightarrow$ 48 elements

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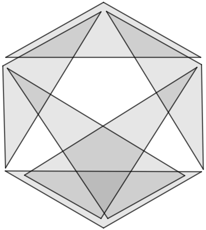
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 3 & 4 & 5 \\ 3 & 6 & 4 & 5 & 2 & 1 \end{matrix} \rightarrow$$



- Distinguishing feature: pair of adjacent double edges
- Stabilizer: $(\mathbb{Z}/2)^2 \rightarrow 8$ elements

Covering Data for Rank 4 over Rank 3

Same as before (tree, strings), but need to identify resultant class.
Rank 3 over rank 2: derangement classes readily identified.

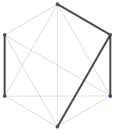
Rank 4 over rank 3:
Extract the string 432615 from $J - I - P_{(12)(3456)}$:

3	4	6	2	$\boxed{1}$	$\boxed{5}$
$\boxed{4}$	6	5	3	2	1
5	$\boxed{3}$	1	$\boxed{6}$	4	2
6	5	$\boxed{2}$	1	3	4

$\rightarrow$

3	5	6	1	4	2
5	6	1	2	3	4
6	4	5	3	2	1

$\rightarrow$



The resulting class is *Ib* (three double edges at a vertex).

Path numbers in rank 4: 322 checks like this.

Outro:

- Clebsch-Gordan coefficients for $SU(2)$: theory of $M(3)$
- CANT 2021: Chu-Vandermonde convolution for finite graded posets
- Syzygies → measure of non-uniqueness of M via B.-v.N.

$$(M, S_1, S_2) \quad \rightarrow \quad \sum(S_1) - \sum(S_2) = 0$$

$$S_i = \{P_\sigma\} \quad \text{(multiset)}$$

(See also R. Stanley's 2014 NYU talk - durer.pdf)

Thank you!

Special thanks to

- CANT Workshop at CUNY
- Emmy Noether Seminar at FAU-Erlangen/Nuremburg
- Lattice Paths, Combinatorics, and Interactions
  (CIRM, June 21-25, 2021) - (online participant only)

References

1. R. Stanley, Enumerative Combinatorics, Volume 1
   (Semi-magic squares, posets)
2. R. Stanley, Algebraic Combinatorics
   (Young lattice, posets with group action, wreath products)
3. D. S. Stones, On the many formulae for the number of Latin rectangles, Electron. J. Combin. 17 (2010)

   (Convenient tables for verification of total path numbers)
4. R. Brualdi, Combinatorial Matrix Classes
   (Sums of permutation matrices and faces of the Birkhoff polytope)

See also references in preprint.