Counting Problems for Lattices

Robert W. Donley, Jr.
(CUNY-QCC)

July 1, 2021

Table of Contents:

Binomial Coefficients

Factorial:

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$$

Binomial Coefficient: $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \ldots (n-k+1)}{k!}$$

When $n < 0$, we can use the second expression.

If $k < 0$, binomial coefficients always equal zero.

Binomial Coefficient

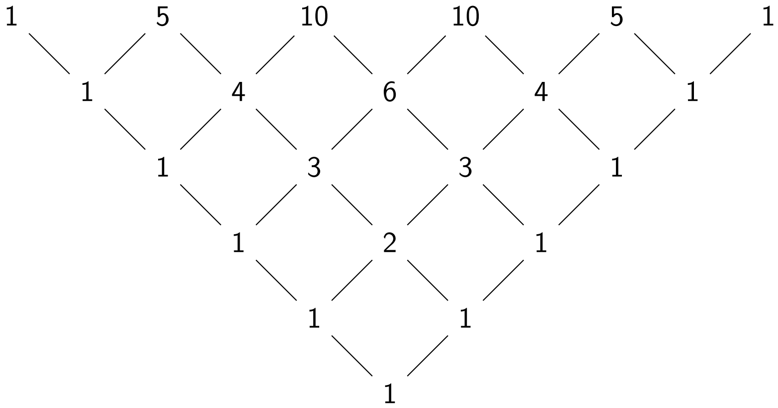
Interpretation: "n choose k":

The number of ways to choose k objects from a set of n objects

1. There are $n!$ ways to order n objects (permutations), say,
$$a_1,\ a_2,\ \ldots, a_n \quad \to \quad n \cdot (n-1) \ldots 1$$
2. A choice of k objects corresponds to placing a partition in the ordering
$$a_1, \ldots, a_k \mid a_{k+1},\ \ldots, a_n.$$
3. To remove dependence on ordering, divide by $k!$ and $(n-k)!$

$$|n - \text{orderings}| \quad = \quad |k - \text{choices}| \cdot |k - \text{orderings}| \cdot |(n-k) - \text{orderings}|$$

Pascal's Triangle

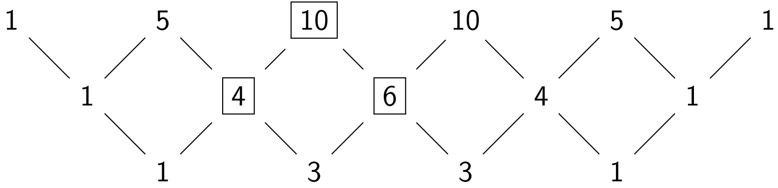


Start at Row 0, then count up.

k-th entry in Row n is $\binom{n}{k}$.

Ex: $\binom{4}{2} = \frac{4!}{2!\,2!} = 6$

Pascal's Identity



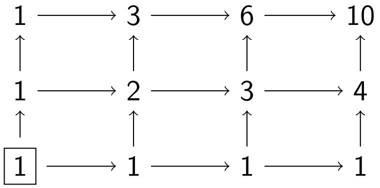
Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Point: To get an entry in the n-th row, add the two below.

Path Counting on an $m \times n$ grid

$(m, n) = (\text{base, height})$



Record the number of directed paths from the origin (SW corner) to any vertex
with **vertex numbers**.

One obtains a fragment of Pascal's Triangle.

Path Counting on an $m \times n$ grid

RURRU

$$
\begin{array}{ccccccc}
1 & & 3 & & 6 & & \boxed{10} \\
 & & & & & & \uparrow \\
1 & & 2 & \longrightarrow & 3 & \longrightarrow & 4 \\
 & & \uparrow & & & & \\
\boxed{1} & \longrightarrow & 1 & & 1 & & 1
\end{array}
$$

$$|Paths : \ SW \to NE| = \binom{m+n}{m}$$

Paths correspond to words of length $m+n$ with m Rs and n Us:

Example: 3×2 grid above. Choose 3 positions from 5 for Rs.

$$\binom{3+2}{3} = 10$$

Partially Ordered Sets

Definition:

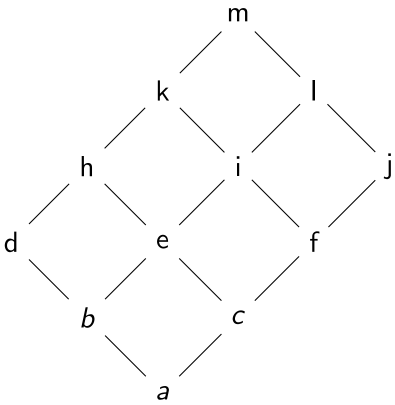
A non-empty set P with binary relation $\leqslant$ is called a

partially ordered set (poset)

if, for all x, y, z in P, the binary relation $\leqslant$ satisfies the following properties

1. Reflexive: $x \leqslant x$,
2. Anti-symmetric: if $x \leqslant y$ and $y \leqslant x$, then $x = y$, and
3. Transitive: if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

Examples: Hasse Diagram of P



$a \leqslant b \leqslant c \leqslant d \leqslant e \leqslant f$

Point: We can recover $\leqslant$ completely from links in diagram.

Finite Graded Posets

$(P, \leqslant)$ finite poset with $\hat{0}$ and $\hat{1}$ (minimum and maximum)

P graded of rank n:

The length of every path from $\hat{0}$ to $\hat{1}$ equals the same n

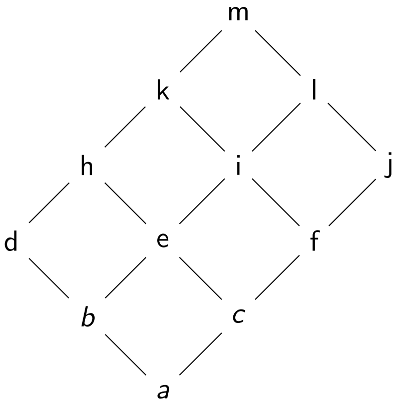
Rank function $\rho: \ P \to \{0, 1, \ldots, \ n\}$

$$\rho(x) = \text{length of any path from } \hat{0} \text{ to } x$$

Rank numbers

$$P_k = \{x \in P \mid \rho(x) = k\} \qquad |P_k| = p_k$$

Finite Graded Posets (Hasse Diagram of P)



$$\hat{0} = a, \qquad \rho(h) = 3, \qquad P_3 = \{h, i, j\}, \qquad p_3 = 3$$

The Vector Space for a Finite Graded Poset with $\hat{0}$ and $\hat{1}$

Use elements of P as a basis (any order)

Definition: The vector space $\mathbb{R}[P]$

Let $\mathbb{R}[P]$ be the vector space over $\mathbb{R}$ with formal basis P;

that is, elements of $\mathbb{R}[P]$ are linear combinations

$$v = \sum_{x \in P} c_x \, x$$

Definition: $x \lessdot y$

For x and y in P, we say y **covers** x if $x \leqslant y$ and no z satisfies $x < z < y$.

The Order-Raising Operator U

Definition: $U : \mathbb{R}[P] \to \mathbb{R}[P]$

For x in P, linearly extend the map

$$Ux = \sum_{x \lessdot y} y$$

Note that Ux is the formal sum of all elements of P directly "above" x.

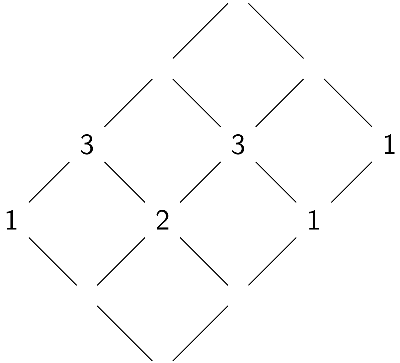
That is, $U|_{P_i} : \mathbb{R}[P_i] \to \mathbb{R}[P_{i+1}]$.

Alternatively, if y is in P_{i+1}, then the coefficient of y in

$$U(\sum_{x \in P} c_x \, x)$$

is the sum of all values c_x just "below" y.

The Order-Raising Operator U

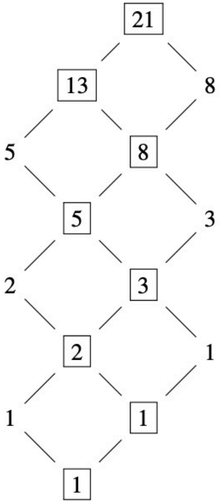


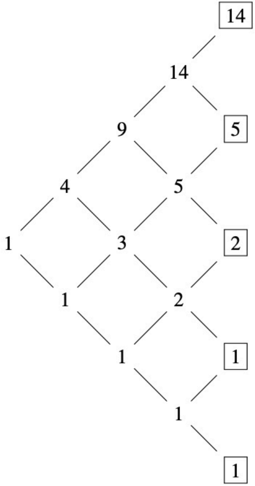
$$U : \mathbb{R}[P_2] \to \mathbb{R}[P_3]$$

$$U(x_1 + 2x_2 + x_3) = 1(x_4) + 2(x_4 + x_5) + 1(x_5 + x_6)$$

$$= 3x_4 + 3x_5 + x_6$$

Fibonacci Numbers / Catalan Numbers



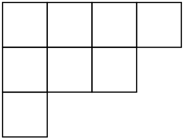
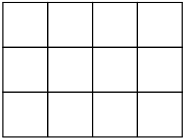


Example: Young Diagrams

Definition: The poset $L(r, s)$

$$P = \{\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r \mid s \geqslant a_r \geqslant \cdots \geqslant a_1 \geqslant 0\}$$

Young diagram: stack r rows of a_i boxes, left justified.

$(4,3,1)$ =  $\leqslant$  = $(4,4,4) = \hat{1}$

Partial order on P: $\mathbf{a} \leqslant \mathbf{b}$ iff $0 \leqslant a_i \leqslant b_i$ *for all i*

$\mathbf{a} \leqslant \mathbf{b}$ iff the diagram for $\mathbf{a}$ fits inside the diagram for $\mathbf{b}$

Example: Young Diagrams

r restricts height, s restricts width

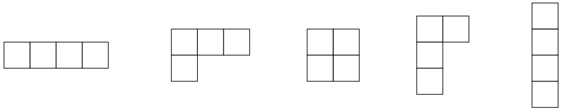
Rank function for P: $\rho(\mathbf{a}) = a_1 + \cdots + a_r$

$$P_k = \{\mathbf{a} \in P \mid a_1 + \cdots + a_r = k\}$$

The rank is just the number of boxes.

Example: $\rho(\mathbf{a}) = 4$ (Partitions of 4)

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$



Example: Young Diagrams

Paths in P from $\hat{0}$ to $\mathbf{a}$ are given by standard tableaux.

Numbers strictly increase along rows and columns

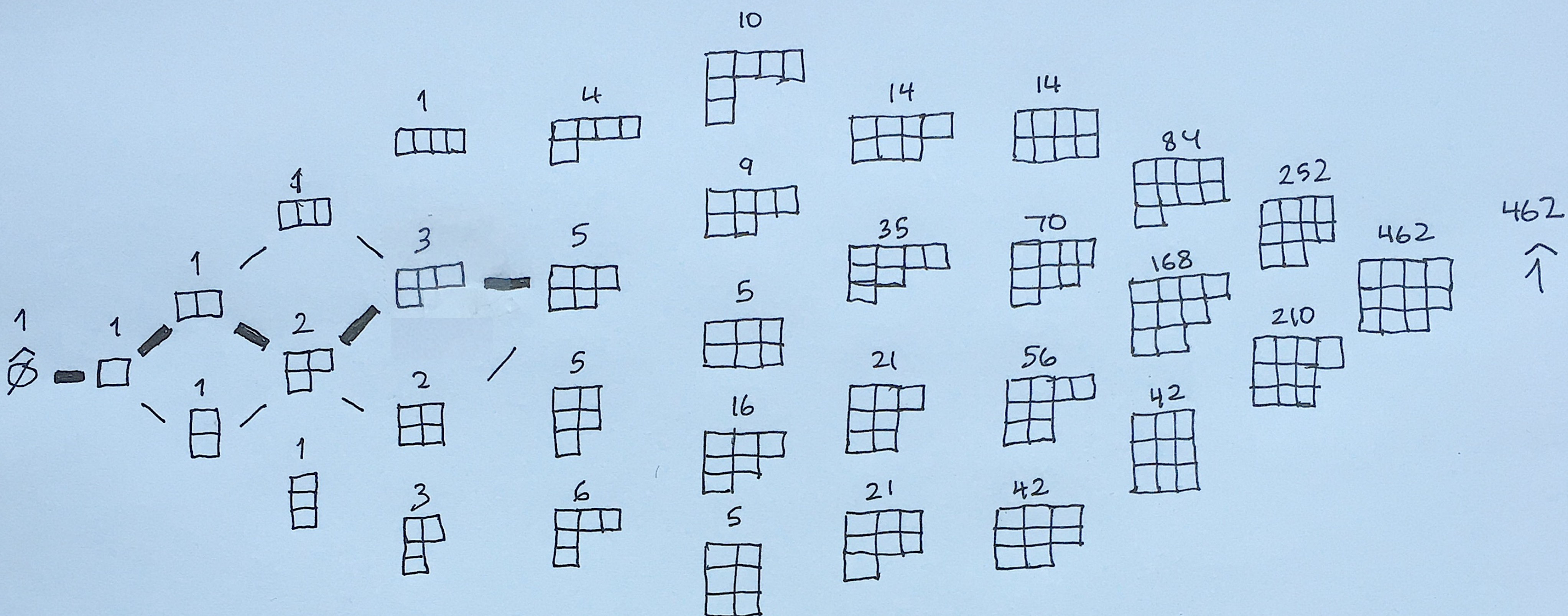
$$\hat{0} \to \begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

Let $f^{\mathbf{a}}$ be the number of standard tableaux of shape $\mathbf{a}$.

$f^{\mathbf{a}}$ is determined using the **hook length formula:**

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} \quad \to \quad f^{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5$$

L(3,4)



6	5	4	3
5	4	3	2
4	3	2	1

$$f^{(4,4,4)} = \frac{12!\ 2}{6!\ 5!\ 4!} = 462$$

Example: Semi-Magic Squares of Size 3

A square matrix is called a **semi-magic square** if

1. entries are integers $\geqslant 0$, and
2. the sum along any row or column is equal to the same number L.

L is called the **line sum** of M.

Example: We consider only the 3×3 case for this talk.

$$M = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}, \qquad L = 5.$$

Examples: $L = 1$

A **permutation matrix** is a square matrix such that there is exactly one 1 in each row and column.

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Algebra for Semi-Magic Squares

1) If $k \geqslant 0$ and M, N are semi-magic squares, so are

$$kM, \qquad M + N$$

with line sums kL_M and $L_M + L_N$, respectively.

2) Any linear combination with $a_i \geqslant 0$ integers

$$a_1 P_1 + \ldots a_6 P_6$$

is a semi-magic square with line sum $a_1 + \cdots + a_6$.

3) In fact, any semi-magic square of size three is of the form in (2). (Induction on L.)

Is $\{P_i\}$ a basis? No.

Solve:

$$\sum a_i P_i = \begin{pmatrix} a_1 + a_6 & a_3 + a_5 & a_2 + a_4 \\ a_2 + a_5 & a_1 + a_4 & a_3 + a_6 \\ a_3 + a_4 & a_2 + a_6 & a_1 + a_5 \end{pmatrix} = 0.$$

Solution:

$$a_1 = a_2 = a_3 = 1, \quad a_4 = a_5 = a_6 = -1,$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Rectangles

Every semi-magic square M can be represented by a rectangle:

$$M \quad \leftrightarrow \quad \mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \quad \leftrightarrow \quad \begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline \end{array}.$$

Here the single relation takes the form

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array}.$$

By repeatedly shifting up, uniquely represented if one of a_4, a_5, a_6 is zero.

Note: line sum $L = a_1 + \cdots + a_6$ is unchanged by shifting 1s

Counting by Line Sum

Question: How many semi-magic squares are there with fixed line sum L? (MacMahon 1916)

$$H_3(L) \;=\; \binom{L+5}{5} - \binom{L+2}{5}$$

First term: put L balls in 6 boxes.

Second term: put $L-3$ balls in 6 boxes

Throw away rectangles of the form: $\quad (L-3)+3 = L$

$$\begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array}.$$

Put L balls in k boxes? $L+(k-1)$ choose $k-1$
(assume 1s at ends) $L = 5, k = 4: \quad \rightarrow \quad 1: \; 01000101 \; :1$

Counting by Orbits

Calculate:

$$H_3(0) = 1, \quad H_3(1) = 6, \quad H_3(2) = 21, \quad H_3(3) = 55, \ldots$$

$\mathbf{H_3(1) = 6}$: Permutation matrices
Start with identity matrix. Apply 6 row permutations.

$\mathbf{H_3(2) = 21}$: either 2+0+0 or 1+1+0 to get $L = 2$.

1	1	0
0	0	0

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

1	0	0
1	0	0

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

2	0	0
0	0	0

$$\rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$6 + 9 + 6 \;=\; 21$$

Symmetries: Wreath Product

At matrix level, the magic square property and line sum are preserved by

1. row permutations,
2. column permutations, and
3. transpose.

At rectangle level, the effect is to

1. switch rows
2. allow permutations in row entries.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}$$

Research:

$M(3)$ forms a graded poset:
Partial ordering (entry-wise for all entries):

$$M \leqslant N \quad \text{if} \quad m_{ij} \leqslant n_{ij}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \leqslant \begin{pmatrix} 2 & 4 & 0 \\ 1 & 2 & 3 \\ 3 & 0 & 3 \end{pmatrix}$$

Rank function $\rho(M) = L$

$M(3,2)$ as rectangles: max entry in matrix is 2 or less

$\begin{array}{|c|c|c|}\hline 2&0&0\\\hline 0&0&0\\\hline\end{array}_{6,1}$ $\begin{array}{|c|c|c|}\hline 2&1&0\\\hline 0&0&0\\\hline\end{array}_{12,3}$ $\begin{array}{|c|c|c|}\hline 2&2&0\\\hline 0&0&0\\\hline\end{array}_{6,6}$

$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline\end{array}_{1,1}$ $\begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&0\\\hline\end{array}_{6,1}$ $\begin{array}{|c|c|c|}\hline 1&1&0\\\hline 0&0&0\\\hline\end{array}_{6,2}$ $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 0&0&0\\\hline\end{array}_{1,12}$ $\begin{array}{|c|c|c|}\hline 2&1&1\\\hline 0&0&0\\\hline\end{array}_{6,36}$ $\begin{array}{|c|c|c|}\hline 2&2&1\\\hline 0&0&0\\\hline\end{array}_{6,150}$ $\begin{array}{|c|c|c|}\hline 2&2&2\\\hline 0&0&0\\\hline\end{array}_{1,900}$

$\begin{array}{|c|c|c|}\hline 1&0&0\\\hline 1&0&0\\\hline\end{array}_{9,2}$ $\begin{array}{|c|c|c|}\hline 1&1&0\\\hline 1&0&0\\\hline\end{array}_{18,6}$ $\begin{array}{|c|c|c|}\hline 1&1&0\\\hline 1&1&0\\\hline\end{array}_{9,24}$

To get to $\begin{array}{|c|c|c|}\hline 3&3&3\\\hline 0&0&0\\\hline\end{array}$: 94,080 paths

To get to $\begin{array}{|c|c|c|}\hline 4&4&4\\\hline 0&0&0\\\hline\end{array}$: 11,988,900 paths

$M(3,2)$ as semi-magic squares:

$$\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\quad\begin{bmatrix}2&0&1\\1&2&0\\0&1&2\end{bmatrix}\quad\begin{bmatrix}2&0&2\\2&2&0\\0&2&2\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\quad\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\quad\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\quad\begin{bmatrix}2&1&2\\2&2&1\\1&2&2\end{bmatrix}\quad\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}$$

$$\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}\quad\begin{bmatrix}1&0&2\\1&2&0\\1&1&1\end{bmatrix}\quad\begin{bmatrix}1&2&1\\1&2&1\\2&0&2\end{bmatrix}$$

Thank you!

For slides, email at rdonley@qcc.cuny.edu