Vandermonde convolution for ranked posets

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Goal: 5 interpretations of Chu-Vandermonde convolution (joining of complementary parts)

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5. The Vector Space $\mathbb{C}P$ and the Order-Raising Operator U
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Doppelgänger (twin stranger):
What happens when a poset meets its dual?

Star Trek Mirror Universe (1967), Harlan Ellison's Shatterday (1980), Timecop (1994), Twin Peaks: The Return (2017)

Chu-Vandermonde Convolution

Chu-Vandermonde Convolution

Fix $m, n \geqslant 0$. Then, for all $0 \leqslant k \leqslant \min(m, n)$,

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}.$$

Interpretation #1:
Suppose

$$X = A \cup B \text{ (disjoint union)}, \qquad |A| = m \quad \text{and} \quad |B| = n.$$

The convolution counts all subsets of X with k elements,
partitioned using i choices from A, $k - i$ choices from B (independent).

Chu-Vandermonde Convolution

Chu-Vandermonde Convolution

Fix $m, n \geqslant 0$. Then, for all $0 \leqslant k \leqslant \min(m, n)$,

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

Interpretation #2:
Extract the coefficient of x^k from each side of

$$(x+1)^m (x+1)^n = (x+1)^{m+n}$$

$$\Big(\sum_i \binom{m}{i} x^i\Big) \quad \Big(\sum_j \binom{n}{j} x^j\Big) = \sum_t \binom{m+n}{t} x^t$$

Chu-Vandermonde Convolution: $m \times n$ grid

$(m, n) = $ (base, height)

$$\begin{array}{ccccccc} 1 & \longrightarrow & 3 & \longrightarrow & 6 & \longrightarrow & 10 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 1 \end{array}$$

Interpretation #3:

Record the number of paths from the origin (SW corner) to any vertex with **vertex numbers**.

Chu-Vandermonde Convolution: $m \times n$ grid

RURRU

$$\begin{tikzcd}
1 & 3 & 6 & 10 \\
1 & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[u] \\
1 \arrow[r] & 1 \arrow[u] & 1 & 1
\end{tikzcd}$$

$$|Paths: \ SW \to NE| = \binom{m+n}{m}$$

Paths correspond to words of length $m+n$ with m Rs and n Us:

Example: 3×2 grid above

$$\binom{3+2}{3} = 10$$

Chu-Vandermonde Convolution: $m \times n$ grid

$$
\begin{tikzcd}
1 \arrow[r] & \boxed{3} \arrow[r] & 6 \arrow[r] & 10 \\
1 \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & \boxed{3} \arrow[r] \arrow[u] & 4 \arrow[u] \\
1 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & \boxed{1} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r] & \boxed{3 * 1} & 1 \arrow[l] \arrow[d] & 1 \arrow[l] \arrow[d] \\
1 \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & \boxed{3 * 2} & 1 \arrow[l] \arrow[d] \\
1 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & \boxed{1 * 1}
\end{tikzcd}
$$

Interpretation #3:

$$|SW \to NE| = \sum_{\boxed{x}} |SW \to x| \cdot |x \to NE|$$

$$= \sum_{\boxed{x}} |SW \to x| \cdot |NE \to x|$$

$$3 + 6 + 1 = 10$$

Finite Graded Posets

$(P, \leqslant)$ finite poset with $\hat{0}$ and $\hat{1}$

P graded of rank n:

The length of every path from $\hat{0}$ to $\hat{1}$ equals the same n

Rank function $\rho: \; P \to \{0, 1, \ldots, \; n\}$

$$\rho(x) = \text{length of any path from } \hat{0} \text{ to } x$$

Rank numbers

$$P_k = \{x \in P \mid \rho(x) = k\} \qquad\qquad |P_k| = p_k$$

Finite Graded Posets

$(P, \leqslant)$ finite graded poset (with $\hat{0}$ and $\hat{1}$)

Dual poset $(P^*, \leqslant_*)$:

$$x \to x^*, \qquad x \leqslant y \iff y^* \leqslant_* x^*$$

Rank function and numbers

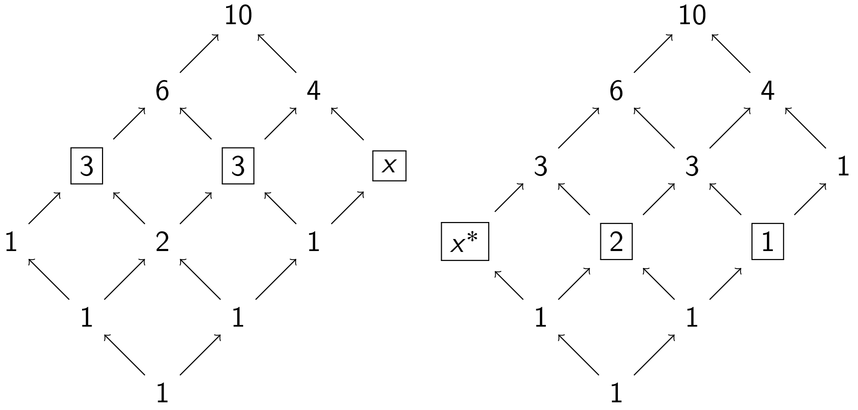
$$\rho_*(x) = n - \rho(x), \qquad (P^*)_k = (P_{n-k})^*, \qquad p_k^* = p_{n-k}$$

Extreme elements in P^*

Minimum: $\hat{1}^*$, Maximum: $\hat{0}^*$

Net effect: turn P upside down

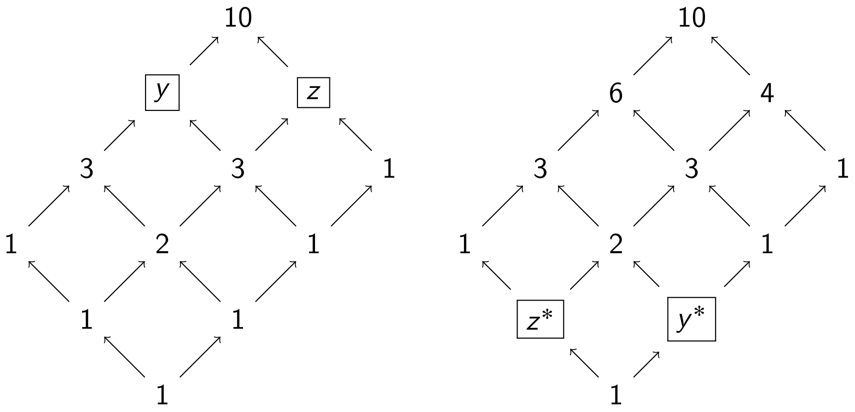
Finite Graded Posets (Hasse Diagrams of P and P^*)



$$|\hat{0} \to \hat{1}| = \sum_{x \in P_3} |\hat{0} \to x| \cdot |\hat{1}^* \to x^*|,$$

$$10 \;=\; 3 \cdot 1 \;+\; 3 \cdot 2 \;+\; 1 \cdot 1$$

Finite Graded Posets $(P$ and $P^*)$



$$|\hat{0} \to \hat{1}| = \sum_{x \in P_4} |\hat{0} \to x| \cdot |\hat{1}^* \to x^*|$$

$$10 = 6 \cdot 1 + 4 \cdot 1$$

Finite Graded Posets $(P \text{ and } P^*)$

$$\begin{tikzcd}
 & & & \boxed{10} & & \\
 & & 6 \arrow[ur] & & 4 \arrow[ul] & \\
 & 3 \arrow[ur] & & 3 \arrow[ul] \arrow[ur] & & 1 \arrow[ul] \\
1 \arrow[ur] & & 2 \arrow[ul] \arrow[ur] & & 1 \arrow[ul] \arrow[ur] & \\
 & 1 \arrow[ul] \arrow[ur] & & 1 \arrow[ul] \arrow[ur] & & \\
 & & 1 \arrow[ul] \arrow[ur] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & 10 & & \\
 & & 6 \arrow[ur] & & 4 \arrow[ul] & \\
 & 3 \arrow[ur] & & 3 \arrow[ul] \arrow[ur] & & 1 \arrow[ul] \\
1 \arrow[ur] & & 2 \arrow[ul] \arrow[ur] & & 1 \arrow[ul] \arrow[ur] & \\
 & 1 \arrow[ul] \arrow[ur] & & 1 \arrow[ul] \arrow[ur] & & \\
 & & \boxed{1} \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

$$|\hat{0} \to \hat{1}| = |\hat{0} \to \hat{1}| \cdot |\hat{1}^* \to \hat{1}^*|$$

$$10 \;=\; 10 \cdot 1$$

Example: Weak compositions. Fix s in $\mathbb{N}$.

Definition: Weak Compositions with maximum s

$$P = \{\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r \mid 0 \leqslant a_i \leqslant s \text{ for all } 1 \leqslant i \leqslant r\}$$

Each **a** determines a rectangular box based at the origin inside the r-cube with side length s.

Partial order on P: $\mathbf{a} \leqslant \mathbf{b}$ iff $0 \leqslant a_i \leqslant b_i$ *for all* i

The box for **a** is contained in the box for **b**.

Example: Weak compositions.

Rank function for P: $\rho(\mathbf{a}) = a_1 + \cdots + a_r$

$$P_k = \{\mathbf{a} \in P \mid a_1 + \cdots + a_r = k\}$$

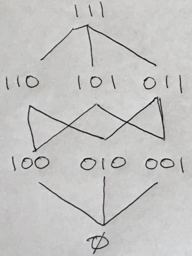
All rectangular boxes with corner on fixed hyperplane.

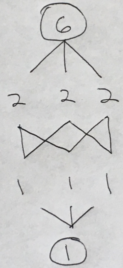
$$\hat{0} = (0, \ldots, 0), \qquad \hat{1} = (s, \ldots, s), \qquad \rho(\hat{1}) = rs.$$

Involution on P: $(a_1, \ldots, a_r)^* = (s - a_1, \ldots, s - a_r)$

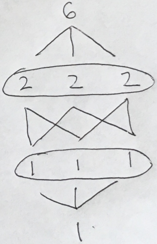
$$(P_k)^* = P_{rs-k}$$

C(3,1)





$1 \cdot 6 = 6$



$1 \cdot 2 + 1 \cdot 2 + 1 \cdot 2 = 6$

Example: Multinomial Convolution

Path counting in a r-cube with side length s:

Theorem (MacMahon?)

Fix $r, s \geqslant 0$. Then, for all $0 \leqslant k \leqslant rs$,

$$\sum_{\substack{\sum a_i = k, \\ 0 \leqslant a_i \leqslant s}} \binom{k}{a_1, \ldots, a_r} \binom{rs-k}{s-a_1, \ldots s-a_r} = \binom{rs}{s, \ldots, s}.$$

Interpretation #2: Extract coefficients of $x_1^s \ldots x_r^s$ in

$$(x_1 + \cdots + x_r)^k (x_1 + \cdots + x_r)^{rs-k} = (x_1 + \cdots + x_r)^{rs}$$

Example: Multinomial Convolution

Generalize: replace $\hat{1} = (s, \ldots, s)$ with $(s_1, .., s_r)$.

Interpret: Counting lattice paths in a rectangular r-box.

Corollary

Fix $r, s_1, \ldots, s_r \geqslant 0$. Then, for all $0 \leqslant k \leqslant rs$,

$$\sum_{\substack{\sum a_i = k, \\ 0 \leqslant a_i \leqslant s_i}} \binom{s_1}{a_1} \cdots \binom{s_r}{a_r} = \binom{s_1 + \cdots + s_r}{k}.$$

This is just **Interpretation #1:**

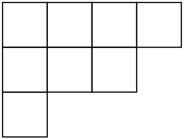
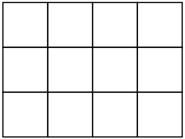
Partition X into $S_1, \ldots, S_r$, and choose k elements of X.

Example: Young Diagrams

Definition: The poset $L(r, s)$

$$P = \{\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r \mid s \geqslant a_r \geqslant \cdots \geqslant a_1 \geqslant 0\}$$

Young diagram: stack r rows of a_i boxes, left justified.

$(4, 3, 1)$ =  $\leqslant$  = $(4, 4, 4) = \hat{1}$

Partial order on P: $\mathbf{a} \leqslant \mathbf{b}$ iff $0 \leqslant a_i \leqslant b_i$ *for all* i

$\mathbf{a} \leqslant \mathbf{b}$ iff the diagram for $\mathbf{a}$ fits inside the diagram for $\mathbf{b}$

Example: Young Diagrams

Rank function for P: $\rho(\mathbf{a}) = a_1 + \cdots + a_r$

$$P_k = \{\mathbf{a} \in P \mid a_1 + \cdots + a_r = k\}$$

Involution on P: $(a_1, \ldots, a_r)^* = (s - a_r, \ldots, s - a_1)$

$$(P_k)^* = P_{rs-k}$$

Remove boxes for λ from $\hat{1} = (s, \ldots, s)$ and rotate.

$$(4,3,1) \quad \to \quad \begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline * & * & * & \\ \hline * & & & \\ \hline\end{array} \quad \to \quad \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \to \quad (3,1,0)$$

Example: Young Diagrams

Paths in P from $\hat{0}$ to λ are given by **standard tableaux**.

Numbers strictly increase along rows and columns

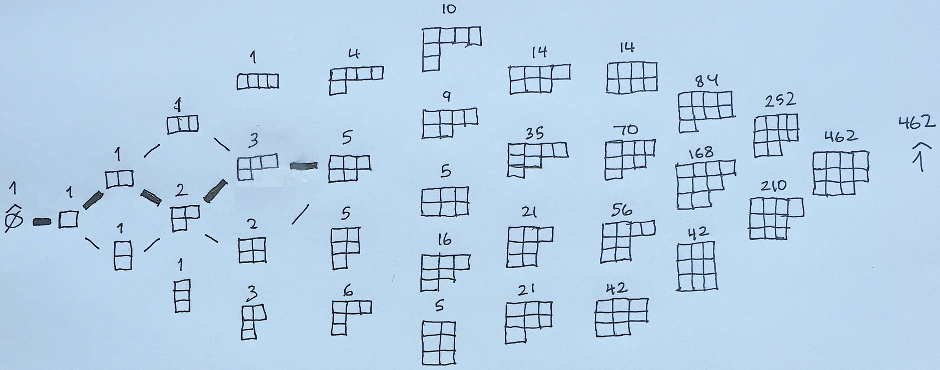
$$\hat{0} \to \begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

Let f^λ be the number of standard tableaux of shape λ.

f^λ is determined using the **hook length formula:**

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} \quad \to \quad f^{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5$$

$L(3,4)$



6	5	4	3
5	4	3	2
4	3	2	1

$$f^{(4,4,4)} = \frac{12!\ 2}{6!\ 5!\ 4!} = 462$$

Example: Young Diagrams

Theorem (Cauchy Identity for Schur Functions - Variant)

Fix $r, s \geqslant 0$. Then, for all $0 \leqslant k \leqslant rs$,

$$\sum_{\substack{\lambda \vdash k \\ \lambda \leqslant (s,\ldots,s)}} f^{\lambda} \, f^{\lambda^*} = f^{(s,\ldots,s)} = C_r(s)$$

where

$$C_r(s) = \frac{(rs)!}{\Pi}$$

and Π is the product of hook lengths in an $r \times s$ rectangle.

$C_r(s)$ are the r-dimensional Catalan numbers.

$r = 2$: Catalan numbers

5	4	3	2
4	3	2	1

$k = 0$: $\quad C_2(s) = \frac{(2s)!}{(s+1)!s!}$

$k = s$: $\quad C_2(s)$ expressed as a sum of $\left\lfloor \frac{s}{2} \right\rfloor + 1$ squares. $(\lambda = \lambda^*)$

The Vector Space for a Finite Graded Poset with $\hat{0}$ and $\hat{1}$

Fix any ordering of elements of P with non-decreasing rank:
$B = \{\hat{0}, \ldots, x, \ldots, \hat{1}\}$

Definition: The vector space $\mathbb{Q}[P]$

Let $\mathbb{Q}[P]$ be the vector space over $\mathbb{Q}$ with formal basis B;

that is, elements of $\mathbb{Q}[P]$ are linear combinations

$$v = \sum_{x \in B} c_x \, x$$

Definition: $x \lessdot y$

For x and y in P, we say y **covers** x if $x \leqslant y$ and no z satisfies $x < z < y$.

The Order-Raising Operator U

Definition: $U : \mathbb{Q}[P] \to \mathbb{Q}[P]$

For x in B, linearly extend the map

$$Ux = \sum_{x \lessdot y} y$$

Note that Ux is the formal sum of all elements of P directly "above" x.

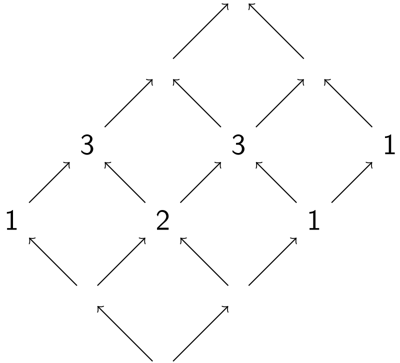
That is, $U|_{P_i} : \mathbb{Q}[P_i] \to \mathbb{Q}[P_{i+1}]$.

Alternatively, if y is in P_{i+1}, then the coefficient of y in

$$U(\sum_{x \in B} c_x \, x)$$

is the sum of all values c_x just "below" y.

The Order-Raising Operator U



$$U : \mathbb{Q}[P_2] \to \mathbb{Q}[P_3]$$

$$U(x_1 + 2x_2 + x_3) = 1(x_4) + 2(x_4 + x_5) + 1(x_5 + x_6)$$

$$= 3x_4 + 3x_5 + x_6$$

The Order-Raising Operator U

Question: Iterating U on $\hat{0}$ counts paths from $\hat{0}$.

What about the rest of $\mathbb{Q}[P]$? No time (JCF), but...

Consider U as a matrix in the basis B. Then

1. U is the (directed) adjacency matrix for the Hasse diagram of P,
2. $u_{ij} = 1$ if $x_j \lessdot x_i$, else 0,
3. U is lower-triangular nilpotent,
4. in fact, $U^n \hat{0} = c\hat{1}$,
5. the non-zero element c of U^n (lower left-hand corner) is the number of paths from $\hat{0}$ to $\hat{1}$.

Interpretation #4:
Vandermonde convolution for a finite graded poset with $\hat{0}$ and $\hat{1}$:

$$U^k \cdot U^{n-k} = U^n$$

Invariant Bilinear Forms (no complex conjugation)

Define a bilinear form on $\mathbb{Q}[P] \times \mathbb{Q}[P^*]$ by extending

$$\langle x, y^* \rangle = \delta_{x,y}$$

If U^* is the order operator for $\mathbb{Q}[P^*]$, then

$$\langle Ux, y^* \rangle = \langle x, U^* y^* \rangle.$$

This gives

Interpretation #5: Convolution as an invariance property.

$$\langle U^k\ \hat{0}, (U^*)^{n-k}\ \hat{1}^* \rangle = \langle U^n\ \hat{0}, \hat{1}^* \rangle$$

Extend to Lie algebra $sl(2,\mathbb{Q}) = Span\{U, D, H\}$?

Peck posets: G.W. PECK, Stanley, Proctor, Robert G. Donnelly

(Graham, West, Purdy, Erdős, Chung, Kleitman)

Thank you!

References

1. W. Fulton, Young Tableaux
2. P. MacMahon, Combinatory Analysis
3. R. Stanley, Enumerative Combinatorics, Volume 1
4. R. Stanley, Algebraic Combinatorics

Honorable mention:
S. Ault and C. Kicey, Counting Lattice Paths Using Fourier Methods (2019)