Binomial Arrays and Generalized Vandermonde Identities

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Binomial Coefficients

Non-negative numerator

For $n \geq 0$ and $0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Negative numerator

For $n \geq 1$ and $k \geq 0$,

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

In all other cases,

$$\binom{n}{k} = 0$$

Catalan Numbers C_n

Segner's Recurrence for C_n

$C_0 = 1$, and, for $n \geq 1$,

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Interpret: $$C_{n+1} = (C_0, C_1, \ldots, C_n) \cdot (C_n, C_{n-1}, \ldots, C_0).$$

Direct Definition

For $n \geq 1$,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n}\binom{2n}{n+1}$$

Catalan Numbers C_n

$$
\begin{aligned}
C_0 &= 1 \\
C_1 &= 1 \cdot 1 = 1 \\
C_2 &= (1,1) \cdot (1,1) = 2 \\
C_3 &= (1,1,2) \cdot (2,1,1) = 5 \\
C_4 &= (1,1,2,5) \cdot (5,2,1,1) = 14 \\
C_5 &= (1,1,2,5,14) \cdot (14,5,2,1,1) = 42
\end{aligned}
$$

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796, \ldots$$

Interpretation

Stanley's List (2015): Over 200 examples as enumeration
Euler: C_n is the number of triangulations of a regular $(n+2)$-gon

C_n is the number of ordered lists with $(n+1)$ $+$s and n $-$s such that all partial sums are positive.

$C_0 : \quad +$

$C_1 : \quad ++-$

$C_2 : \quad +++--, \; ++-+-$

$C_3 : \quad ++++---, \quad +++-+--,$
$\quad +++--+-, \quad ++-+-+-, \quad ++-++--$

Shapiro's Catalan Triangle

Shapiro's Catalan triangle for $B_{n,k}$						
$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	1				
3	5	4	1			
4	$\boxed{14}$	$\boxed{14}$	$\boxed{6}$	1		
5	42	$\boxed{48}$	27	8	1	
6	132	165	110	44	10	1

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k}$$

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}$$

Shapiro's Catalan Triangle

Shapiro's Catalan triangle for $B_{n,k}$						
$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	1				
3	5	4	1			
4	14	14	6	1		
5	42	48	27	8	1	
6	132	165	110	44	10	1

Shapiro (1976):
Dot product of any row with itself or two (adjacent) rows is another Catalan number

Examples:

$$(2,1,0,0)\cdot(14,14,6,1) = 28+14+0+0 = 42,$$

$$(14,14,6,1)\cdot(14,14,6,1) = 196+196+36+1 = 429.$$

Kirillov-Melnikov's Catalan triangle (1996)

$$\begin{bmatrix} 1 & \boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 & \cdots \\ -1 & 0 & 1 & \boxed{2} & 3 & \boxed{4} & 5 & \boxed{6} & 7 & \cdots \\ 0 & -1 & -1 & 0 & 2 & \boxed{5} & 9 & \boxed{14} & 20 & \cdots \\ 0 & 0 & -1 & -2 & -2 & 0 & 5 & \boxed{14} & 28 & \cdots \\ 0 & 0 & 0 & -1 & -3 & -5 & -5 & 0 & 14 & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Rule:

1. $(1, -1, 0, 0, \dots)$ down left column,
2. 1s along top row, and
3. capital L-summation to progress to the right.

Dot Product Rule

$$\begin{bmatrix} \boxed{1} & 1 & 1 & \boxed{1} & 1 & \boxed{1} & 1 & 1 & \boxed{1} & \cdots \\ \boxed{-1} & 0 & 1 & \boxed{2} & 3 & \boxed{4} & 5 & 6 & \boxed{7} & \cdots \\ \boxed{0} & -1 & -1 & \boxed{0} & 2 & \boxed{5} & 9 & 14 & \boxed{20} & \cdots \\ \boxed{0} & 0 & -1 & \boxed{-2} & -2 & \boxed{0} & 5 & 14 & \boxed{28} & \cdots \\ \boxed{0} & 0 & 0 & \boxed{-1} & -3 & \boxed{-5} & -5 & 0 & \boxed{14} & \cdots \\ \boxed{0} & 0 & 0 & \boxed{0} & -1 & \boxed{-4} & -9 & -14 & \boxed{-14} & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 & -1 & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

To recover/extend Shapiro's formulas,

1. columns are skew-palindromes, and
2. use convolution (Segner's Rule) to align correctly.

$$(1,\ 3,\ 2, -2, -3, -1) \cdot (-1, -3, -2,\ 2,\ 3,\ 1) = -2(14)$$

$$(1,\ 2,\ 0, -2, -1, 0) \cdot (-4, -5, 0,\ 5,\ 4,\ 1) = -2(14)$$

Shifting Columns for Dot Product

Start with any column; convolution gives $-2C_n$

Unchanged if we telescope inwards or outwards

$$(1,\ 3,\ 2, -2, -3, -1) \cdot (-1, -3, -2,\ 2,\ 3,\ 1) = -28$$

$$(1,\ 2,\ 0, -2, -1, 0) \cdot (-4, -5, 0,\ 5,\ 4,\ 1) = -28$$

$$(1,\ 1,\ -1, -1, 0, 0) \cdot (-9, -5, -1,\ 1,\ 5,\ 9) = -28$$

$$(1,\ 0,\ -1, 0, 0, 0) \cdot (-14, 0, 14,\ 14,\ 6,\ 1) = -28$$

$$(1,\ -1,\ 0, 0, 0, 0) \cdot (-14, 14, 0,\ 0,\ 0,\ 0) = -2(14)$$

Point of talk: Explain this phenomenon in a general setting.

Pascal's Triangle

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 0 & 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & \cdots \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & 15 & 35 & 70 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 21 & 28 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Rule:

1. $(1, 0, 0, \ldots)$ down left column,
2. 1s along top row, and
3. capital L-summation to progress to the right.

Chu-Vandermonde Convolution

$$\begin{bmatrix} \boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 & 1 & \cdots \\ \boxed{0} & 1 & \boxed{2} & 3 & \boxed{4} & 5 & \boxed{6} & 7 & 8 & \cdots \\ \boxed{0} & 0 & \boxed{1} & 3 & \boxed{6} & 10 & \boxed{15} & 21 & 28 & \cdots \\ \boxed{0} & 0 & \boxed{0} & 1 & \boxed{4} & 10 & \boxed{20} & 35 & 56 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & 15 & 35 & 70 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 21 & 28 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\begin{aligned} (1,\ 3,\ 3,1)\cdot(1,3,3,1) \quad &= (1,\ 2,\ 1,\ 0)\cdot(4,\ 6,\ 4,\ 1) \\ &= (1,\ 1,\ 0,0)\cdot(10,10,5,\ 1) \\ &= (1,\ 0,\ 0,\ 0)\cdot(20,\ 15,\ 6,\ 1) \\ &= 20. \end{aligned}$$

Chu-Vandermonde Convolution

This is just the famous Chu-Vandermonde convolution

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}.$$

Suppose $X = A \cup B$ is a disjoint union with $|A| = m$ and $|B| = n$.

If we choose k elements from X, the summation expresses this choice with respect to independent choices from A and B.

In fact, one may allow negative m or n.

Lagrange:

$$\binom{2n}{k} = \sum_{i=0}^{k} \binom{n}{i}^2.$$

Generalized Binomial Transform

$$\{a_i\}_{i=0}^{\infty} \to p(x) = \sum_{i=0}^{\infty} a_i x^i$$

Generalized Binomial Transform

$$B^n a_k = a_{k,n} = \sum_{i=0}^{n} \binom{n}{i} a_{k-i}$$

$$\sum_{k=0}^{\infty} B^n a_k \, x^k = (1+x)^n \sum_{i=0}^{\infty} a_i x^i$$

Usual binomial transform: $B^n a_n = a_{n,n} = \sum_{i=0}^{n} \binom{n}{i} a_{n-i}$

Pascal's Recurrence

$$n = 1 : Ba_k = a_{k-1} + a_k$$
$$n = 2 : B^2 a_k = a_{k-2} + 2a_{k-1} + a_k$$
$$n = 3 : B^3 a_k = a_{k-3} + 3a_{k-2} + 3a_{k-1} + a_k$$

Pascal's Recurrence (Capital L)

$$B^{n+1} a_k = B^n a_k + B^n a_{k-1}$$

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Matrix Implementation

1. Fourth quadrant matrix
2. $\{a_i\}$ down first column, a_0 along first row
3. Pascal's recurrence: Capital-L summation

$$\begin{bmatrix} a_0 & a_0 & a_0 & a_0 & a_0 & \cdots \\ a_1 & a_0 + a_1 & 2a_0 + a_1 & 3a_0 + a_1 & 4a_0 + a_1 & \cdots \\ a_2 & a_1 + a_2 & a_0 + 2a_1 + a_2 & 3a_0 + 3a_1 + a_2 & 6a_0 + 4a_1 + a_2 & \cdots \\ a_3 & a_2 + a_3 & a_1 + 2a_2 + a_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$B^n a_k$: row $k+1$, column $n+1$ (first row and column indexed to 0)

Pascal's triangle

$$a_i = (1,\ 0,\ 0,\ \ldots) \qquad \to \qquad B^n a_k = \binom{n}{k}$$

$$PT = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & 6 & \cdots \\ 0 & 0 & 1 & 3 & 6 & 10 & \boxed{15} & \cdots \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

1. $\uparrow$: Sums to 2^n
2. $\nearrow$: Sums to F_n (Fibonacci numbers)
3. Hockey Stick Summation :

$$1 + 2 + 3 + 4 + 5 = 15$$

Catalan Number Trapezoid

Next simplest initial condition is $(1, 1, 0, 0, \dots)$, which is Pascal's triangle, shifted to left by 1.

$$a_i = (1,\ -1, 0, 0, \dots), \qquad B^n a_i = \binom{n}{i} - \binom{n}{i-1}$$

$$CT = \begin{bmatrix} \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -1 & 0 & \boxed{1} & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 0 & -1 & -1 & 0 & \boxed{2} & 5 & 9 & 14 & 20 & \cdots \\ 0 & 0 & -1 & -2 & -2 & 0 & \boxed{5} & 14 & 28 & \cdots \\ 0 & 0 & 0 & -1 & -3 & -5 & -5 & 0 & \boxed{14} & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$CT = PT - \begin{bmatrix} 0 & 0 & \cdots \\ PT & & \end{bmatrix}$$

Catalan Number Trapezoid

$$CT = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 0 & -1 & -1 & 0 & 2 & 5 & 9 & 14 & 20 & \cdots \\ 0 & 0 & -1 & -2 & -2 & 0 & 5 & 14 & 28 & \cdots \\ 0 & 0 & 0 & -1 & -3 & -5 & -5 & 0 & 14 & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$a_i = (1, -1, 0, 0, \dots) \to p(x) = 1 - x$$

$$\begin{aligned} (1-x)(1+x) &= 1 + 0x - x^2 \\ (1-x)(1+x)^2 &= 1 + x - x^2 - x^3 \\ (1-x)(1+x)^3 &= 1 + 2x + 0x^2 - 2x^3 - x^4 \\ (1-x)(1+x)^4 &= 1 + 3x + 2x^2 - 2x^3 - 3x^4 - x^5 \end{aligned}$$

Extension to Left

Progress to right: multiply by $(1+x)$

Progress to left: divide by $(1+x)$, or multiply by $\sum_{i=0}^{\infty}(-1)^i x^i$

(locally finite if $p(x)$ is power series)

Net effect:

$$B^{-1}a_k = a_k - a_{k-1} + a_{k-2} - a_{k-3} + \cdots \pm a_0.$$

Hockey Stick Rule

To progress to left, alternating sum to top line

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \boxed{1} & 1 & 1 & \cdots \\ -1 & 0 & 1 & 2 & 3 & 4 & \boxed{5} & 6 & 7 & \cdots \\ 0 & -1 & -1 & 0 & 2 & 5 & \boxed{9} & 14 & 20 & \cdots \\ 0 & 0 & -1 & -2 & -2 & 0 & \boxed{5} & 14 & 28 & \cdots \\ 0 & 0 & 0 & -1 & -3 & \boxed{\text{-5}} & \boxed{\text{-5}} & 0 & 14 & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$-5 = (-5) - 5 + 9 - 5 + 1$$

Extended Pascal's Triangle

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & \boxed{1} & 1 & \cdots \\ -4 & \boxed{-3} & -2 & -1 & & 1 & 2 & \boxed{3} & 4 & \cdots \\ 10 & 6 & \boxed{3} & 1 & & & 1 & \boxed{3} & 6 & \cdots \\ -20 & -10 & -4 & \boxed{-1} & & & & \boxed{1} & 4 & \cdots \\ 35 & 15 & 5 & 1 & & & & & 1 & \cdots \\ -56 & -21 & -6 & -1 & & & & & & \cdots \\ \cdots & \cdots & \cdots & \cdots & & & & & & \end{bmatrix}$$

Extended Catalan Trapezoid

-5	-4	-3	-2	-1	0	1	2	3	4	5
1	1	1	1	1	1	1	1	1	1	1
-6	-5	-4	-3	-2	-1	0	1	2	3	4
20	14	9	5	2		-1	-1	0	2	5
-50	-30	-16	-7	-2			-1	-2	-2	0
105	55	25	9	2				-1	-3	-5
-196	-91	-36	-11	-2					-1	-4
336	140	49	13	2						-1

Binomial Array

Definition

Let $\{a_i\}_{i=0}^{\infty}$ be a sequence. To construct the **binomial array** $B(a_i)$,

1. all values in the top line (row zero) are set to a_0,
2. the value in the center column (column zero) and i-th row is a_i, and
3. fill in the lower half plane using Pascal's Recurrence to the right and left.

For $k \geq 0$ and all n in $\mathbb{Z}$, the (k, n)-th entry of $B(a_i)$ is $B^n a_k$.

For convenience, we may denote by $B(p(x))$ if $p(x) = \sum_{i=0}^{\infty} a_i x^i$.

Example: Clebsch-Gordan Hexagon

-3	-2	-1	0	1	2	3	4	5	6	7
10	10	10	10	10	10	10	10	10	10	10
-48	-38	-28	-18	-8	2	12	22	32	42	52
132	84	46	18	0	-8	-6	6	28	60	102
-272	-140	-56	-10	8	8	0	-6	0	28	88
468	196	56		-10	-2	6	6	0	0	28
-720	-252	-56			-10	-12	-6	0	0	0
1028	308	56				-10	-22	-28	-28	-28

Return to Vandermonde Convolution and Column Shifting

Discrete convolution

If a_i and b_i are sequences, we define a new sequence, the **discrete convolution** (or Cauchy product) by

$$(a * b)_n = \sum_{i+j=n} a_i b_j = \sum_{i=0}^{n} a_i b_{n-i}.$$

Alternatively, $(a * b)_n$ is the coefficient c_n in the power series product

$$\sum_{i=0}^{\infty} c_i x^i = (\sum_{j=0}^{\infty} a_j x^j)(\sum_{k=0}^{\infty} b_k x^k).$$

Frankel's Theorem (1958)

Theorem (Generalized Vandermonde Convolution)

If a_i and b_j are sequences, then, for all n in $\mathbb{Z}$,

$$(a * b)_k = (B^n a * B^{-n} b)_k.$$

Interpretation if $a_i = b_i$

1. construct $B(a_i)$,
2. section off any rectangle from the top line,
3. the convolution of the left and right hand sides is unchanged under telescoping.

Example:

-5	-4	-3	-2	-1	0	1	2	3	4	5
2	2	2	2	2	2	2	2	2	2	2
-11	-9	-7	-5	-3	-1	1	3	5	7	9
35	24	15	8	3		-1	0	3	8	15
-85	-50	-26	-11	-3		0	-1	-1	2	10
175	90	40	14	3		0	0	-1	-2	0

$$\begin{aligned} (2,\ 1,\ -1)\cdot(-1,1,2) &= (2,\ -1,\ 0)\cdot(0,\ 3,\ 2) \\ &= (2,\ -3,3)\cdot(3,5,2) \\ &= (2,-5,8)\cdot(8,\ 7,\ 2) \\ &= -3. \end{aligned}$$

Proof

Suppose $p(x)$ and $q(x)$ correspond to a_i and b_j, respectively.

Define $[x^k]p(x) = a_k$. Then $(a * b)_k = [x^k](p(x)q(x))$.

$$\begin{aligned}(B^n a * B^{-n} b)_k &= [x^k]((1+x)^n p(x)(1+x)^{-n} q(x)) \\ &= [x^k](p(x)q(x)) \\ &= (a * b)_k\end{aligned}$$

Hockey Stick from Capital-L

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 7 & 8 & 9 & 10 & 11 \\ 21 & 28 & 36 & 45 & 55 \\ 35 & 56 & 84 & \boxed{120} & 165 \\ 49 & 84 & 140 & \boxed{224} & \boxed{344} \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 7 & 8 & 9 & 10 & 11 \\ 21 & 28 & \boxed{36} & 45 & 55 \\ 35 & 56 & \boxed{84} & 120 & 165 \\ 49 & 84 & 140 & \boxed{224} & \boxed{344} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 7 & \boxed{8} & 9 & 10 & 11 \\ 21 & \boxed{28} & 36 & 45 & 55 \\ 35 & 56 & \boxed{84} & 120 & 165 \\ 49 & 84 & 140 & \boxed{224} & \boxed{344} \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 1 & 1 & 1 & 1 \\ \boxed{7} & 8 & 9 & 10 & 11 \\ 21 & \boxed{28} & 36 & 45 & 55 \\ 35 & 56 & \boxed{84} & 120 & 165 \\ 49 & 84 & 140 & \boxed{224} & \boxed{344} \end{bmatrix}.$$

Hockey Stick from Capital-L

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 15 & 21 & 28 & \boxed{36} & 45 & 55 \\ 20 & 35 & 56 & \boxed{84} & \boxed{120} & 165 \\ 29 & 49 & 84 & 140 & 224 & 344 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 15 & 21 & \boxed{28} & \boxed{36} & 45 & 55 \\ 20 & 35 & \boxed{56} & 84 & \boxed{120} & 165 \\ 29 & 49 & 84 & 140 & 224 & 344 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 15 & \boxed{21} & \boxed{28} & \boxed{36} & 45 & 55 \\ 20 & \boxed{35} & 56 & 84 & \boxed{120} & 165 \\ 29 & 49 & 84 & 140 & 224 & 344 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ \boxed{15} & \boxed{21} & \boxed{28} & \boxed{36} & 45 & 55 \\ \boxed{20} & 35 & 56 & 84 & \boxed{120} & 165 \\ 29 & 49 & 84 & 140 & 224 & 344 \end{bmatrix}$$

Hockey Stick from Capital-L

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 10 & 11 \\ 36 & 45 & 55 \\ \boxed{84} & 120 & 165 \\ \boxed{140} & \boxed{224} & 344 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 9 & 10 & 11 \\ \boxed{36} & 45 & 55 \\ 84 & \boxed{120} & 165 \\ \boxed{140} & \boxed{224} & 344 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \boxed{9} & 10 & 11 \\ 36 & \boxed{45} & 55 \\ 84 & \boxed{120} & 165 \\ \boxed{140} & \boxed{224} & 344 \end{bmatrix}.$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 9 & \boxed{10} & 11 \\ 36 & \boxed{45} & 55 \\ 84 & \boxed{120} & 165 \\ \boxed{140} & \boxed{224} & 344 \end{bmatrix}, \begin{bmatrix} 1 & \boxed{1} & 1 \\ 9 & \boxed{10} & 11 \\ 36 & \boxed{45} & 55 \\ 84 & \boxed{120} & 165 \\ \boxed{140} & \boxed{224} & 344 \end{bmatrix}$$

Near-zero Sequences

0	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
-1	0	**1**	2	3	4	5	6	7	8	9
	-1	**-1**	0	2	5	9	14	20	27	35
		-1	-2	-2	0	5	14	28	48	75
			-1	-3	-5	-5	0	14	42	90
				-1	-4	-9	-14	-14	0	42
					-1	-5	-14	-28	-42	-42
						-1	-6	-20	-48	-90
							-1	-7	-27	-75
								-1	-8	-35
									-1	-9

Catalan Numbers: 1, 1, 2, 5, 14, 42, 132, 429, . . .

Near-zero Sequences

0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
-2	-1	0	1	2	3	4	5	6	7	8	9	10
0	-2	-3	-3	-2	0	3	7	12	18	25	33	42
0	0	-2	-5	-8	-10	-10	-7	0	12	30	55	88
0	0	0	-2	-7	-15	-25	-35	-42	-42	-30	0	55

0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
-3	-2	-1	0	1	2	3	4	5	6	7	8	9
0	-3	-5	-6	-6	-5	-3	0	4	9	15	22	30
0	0	-3	-8	-14	-20	-25	-28	-28	-24	-15	0	22
0	0	0	-3	-11	-25	-45	-70	-98	-126	-150	-165	-165

$$a_i = (r, -s, 0, 0, \dots)$$

$$C_t^{(r,s)} = \frac{1}{t} \; \frac{(rt+st)!}{(rt+1)!\;(st-1)!}$$

$r = s = 1$: Catalan numbers

$s = 1$: Fuss-Catalan numbers

$r = 1$: Related to convolutional codes

Near-zero Sequences

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
-18	-8	2	12	22	32	42	52	62	72	82	92	102	112	122
18	0	-8	-6	6	28	60	102	154	216	288	370	462	564	676
-10	8	8	0	-6	0	2	88	190	344	560	848	1218	1680	2244
0	-10	-2	6	6	0	0	28	116	306	650	1210	2058	3276	4956
0	0	-10	-12	-6	0	0	0	28	144	450	1100	2310	4368	7644
0	0	0	-10	-22	-28	-28	-28	-28	0	144	594	1694	4004	8372
0	0	0	0	-10	-32	-60	-88	-116	-144	-144	0	594	2288	6292
0	0	0	0	0	-10	-42	-102	-190	-306	-450	-594	-594	0	2288
0	0	0	0	0	0	-10	-52	-154	-344	-650	-1100	-1694	-2288	-2288

a_i : Clebsch-Gordan coefficient condition

$$m \text{ odd}, \qquad 1 \leq k' \leq \frac{m-1}{2}$$

$$C_t(m,k') = (m-k')\frac{\binom{m-k'-1}{k'}\binom{t+2k'-m}{k'}}{\binom{t+k'+1}{k'}}C_t,$$

Thank You!

m=6 n=12 k=4

15	15	15	15	15	15	15	15	15	0	0
-90	-75	-60	-45	-30	-15	0	15	30	45	0
270	180	105	45	0	-30	-45	-45	-30	0	45
-495	-225	-45	60	105	105	75	30	-15	-45	-45
495	0	-225	-270	-210	-105	0	75	105	90	45
0	495	495	270	0	-210	-315	-315	-240	-135	-45
0	0	495	990	1260	1260	1050	735	420	180	45

m=12 n=6 k=4

495	495	495	0	0	0	0	0	0	0	0
-495	0	495	990	0	0	0	0	0	0	0
270	-225	-225	270	1260	0	0	0	0	0	0
-90	180	-45	-270	0	1260	0	0	0	0	0
15	-75	105	60	-210	-210	1050	0	0	0	0
0	15	-60	45	105	-105	-315	735	0	0	0
0	0	15	-45	0	105	0	-315	420	0	0
0	0	0	15	-30	-30	75	75	-240	180	0
0	0	0	0	15	-15	-45	30	105	-135	45
0	0	0	0	0	15	0	-45	-15	90	-45
0	0	0	0	0	0	15	15	-30	-45	45
0	0	0	0	0	0	0	15	30	0	-45
0	0	0	0	0	0	0	0	15	45	45

m=6 n=10 k=2

15	15	15	15	15	15	15	15	15	0	0	0	0
-45	-30	-15	0	15	30	45	60	75	90	0	0	0
45	0	-30	-45	-45	-30	0	45	105	180	270	0	0
0	45	45	15	-30	-75	-105	-105	-60	45	225	495	0
0	0	45	90	105	75	0	-105	-210	-270	-225	0	495
0	0	0	45	135	240	315	315	210	0	-270	-495	-495
0	0	0	0	45	180	420	735	1050	1260	1260	990	495

m=10 n=10 k=5

252	252	252	252	252	252	0	0	0	0	0
-756	-504	-252	0	252	504	756	0	0	0	0
1176	420	-84	-336	-336	-84	420	1176	0	0	0
-1176	0	420	336	0	-336	-420	0	1176	0	0
756	-420	-420	0	336	336	0	-420	-420	756	0
-252	504	84	-336	-336	0	336	336	-84	-504	252
0	-252	252	336	0	-336	-336	0	336	252	-252
0	0	-252	0	336	336	0	-336	-336	0	252
0	0	0	-252	-252	84	420	420	84	-252	-252
0	0	0	0	-252	-504	-420	0	420	504	252
0	0	0	0	0	-252	-756	-1176	-1176	-756	-252

252	252	252	252	252	252					
-756	-504	-252	0	252	504	756				
1176	420	-84	-336	-336	-84	420	1176			
-1176	0	420	336	0	-336	-420	0	1176		
756	-420	-420	0	336	336	0	-420	-420	756	
-252	504	84	-336	-336	0	336	336	-84	-504	252
	-252	252	336	0	-336	-336	0	336	252	-252
		-252	0	336	336	0	-336	-336	0	252
			-252	-252	84	420	420	84	-252	-252
				-252	-504	-420	0	420	504	252
					-252	-756	-1176	-1176	-756	-252

m=16 n=16 k=8

12870	12870	12870	12870	12870	12870	12870	12870	12870	0	0	0	0	0	0	0	0
-57915	-45045	-32175	-19305	-6435	6435	19305	32175	45045	57915	0	0	0	0	0	0	0
135135	77220	32175	0	-19305	-25740	-19305	0	32175	77220	135135	0	0	0	0	0	0
-212355	-77220	0	32175	32175	12870	-12870	-32175	-32175	0	77220	212355	0	0	0	0	0
245025	32670	-44550	-44550	-12375	19800	32670	19800	-12375	-44550	-44550	32670	245025	0	0	0	0
-212355	32670	65340	20790	-23760	-36135	-16335	16335	36135	23760	-20790	-65340	-32670	212355	0	0	0
135135	-77220	-44550	20790	41580	17820	-18315	-34650	-18315	17820	41580	20790	-44550	-77220	135135	0	0
-57915	77220	0	-44550	-23760	17820	35640	17325	-17325	-35640	-17820	23760	44550	0	-77220	57915	0
12870	-45045	32175	32175	-12375	-36135	-18315	17325	34650	17325	-18315	-36135	-12375	32175	32175	-45045	12870
0	12870	-32175	0	32175	19800	-16335	-34650	-17325	17325	34650	16335	-19800	-32175	0	32175	-12870
0	0	12870	-19305	-19305	12870	32670	16335	-18315	-35640	-18315	16335	32670	12870	-19305	-19305	12870
0	0	0	12870	-6435	-25740	-12870	19800	36135	17820	-17820	-36135	-19800	12870	25740	6435	-12870
0	0	0	0	12870	6435	-19305	-32175	-12375	23760	41580	23760	-12375	-32175	-19305	6435	12870
0	0	0	0	0	12870	19305	0	-32175	-44550	-20790	20790	44550	32175	0	-19305	-12870
0	0	0	0	0	0	12870	32175	32175	0	-44550	-65340	-44550	0	32175	32175	12870
0	0	0	0	0	0	0	12870	45045	77220	77220	32670	-32670	-77220	-77220	-45045	-12870
0	0	0	0	0	0	0	0	12870	57915	135135	212355	245025	212355	135135	57915	12870