Central Zeros of Clebsch-Gordan Coefficients

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Binomial coefficient

For $n \geq 0$,

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & 0 \leq k \leq n \\ 0 & k < 0, k > n \end{cases}$$

Shifted Binomial Transform

$$\{a_i\}_{i=0}^{\infty} \to \sum_{i=0}^{\infty} a_i x^i$$

Shifted Binomial Transform

$$b_i^n = \sum_{k=0}^{n} \binom{n}{k} a_{i-k}$$

$$\sum_{i=0}^{\infty} b_i^n x^i = (1+x)^n \sum_{i=0}^{\infty} a_i x^i$$

Usual binomial transform: $b_n^n = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}$

Pascal's Triangle

$$n = 1 : b_i^1 = a_{i-1} + a_i$$
$$n = 2 : b_i^2 = a_{i-2} + 2a_{i-1} + a_i$$
$$n = 3 : b_i^3 = a_{i-3} + 3a_{i-2} + 3a_{i-1} + a_i$$

Pascal's Recurrence

$$b_i^{n+1} = b_i^n + b_{i-1}^n$$

Pascal's Identity

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$$

Matrix Implementation

$1D$ cellular automata

1. Fourth quadrant matrix
2. $\{a_i\}$ down first column, a_0 along first row
3. Pascal's recurrence: Capital-L summation

$$\begin{bmatrix} a_0 & a_0 & a_0 & a_0 & a_0 & \cdots \\ a_1 & a_0+a_1 & 2a_0+a_1 & 3a_0+a_1 & 4a_0+a_1 & \cdots \\ a_2 & a_1+a_2 & a_0+2a_1+a_2 & 3a_0+3a_1+a_2 & 6a_0+4a_1+a_2 & \cdots \\ a_3 & a_2+a_3 & a_1+2a_2+a_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

b_i^n: row $i+1$, column $n+1$

Pascal's triangle

$$a_i = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} \qquad \rightarrow \qquad b_i^n = \binom{n}{i}$$

$$PT = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & 6 & \cdots \\ 0 & 0 & \boxed{1} & 3 & 6 & 10 & \boxed{15} & \cdots \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

1. $\uparrow$: Sums to 2^n
2. $\nearrow$: Sums to F_n (Fibonacci numbers)
3. Hockey Stick Summation (with hooking):

$$(1) + (2+3+4+5) = 15$$

Example

$$a_0 = 3,\ \ a_1 = -6,\ \ a_2 = 6, a_i = 0\ (i > 2)$$

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 & \cdots \\ -6 & -3 & 0 & 3 & 6 & \cdots \\ 6 & 0 & -3 & -3 & 0 & \cdots \\ 0 & 6 & 6 & 3 & 0 & \cdots \\ 0 & 0 & 6 & 12 & 15 & \cdots \end{bmatrix}$$

$$= 3\ PT - 6 \begin{bmatrix} 0 & 0 & \cdots \\ PT & & \end{bmatrix} + 6 \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ PT & & \end{bmatrix}$$

Example

$$a_0 = 3,\ a_1 = -6,\ a_2 = 6, a_i = 0\ (i > 2)$$

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 & \cdots \\ \boxed{-6} & \boxed{-3} & \boxed{0} & \boxed{3} & 6 & \cdots \\ \boxed{6} & 0 & -3 & -3 & \boxed{0} & \cdots \\ 0 & 6 & 6 & 3 & 0 & \cdots \\ 0 & 0 & 6 & 12 & 15 & \cdots \end{bmatrix}$$

1. $\uparrow$: Sums to $(3 - 6 + 6)2^n$
2. $\nearrow$: Sums to $6F_n - 6F_{n+1} + 3F_{n+2}$ (Fibonacci numbers)
3. Hockey Stick Summation: (6) + ((-6)+(-3)+0 +3) = 0

Example

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 & \cdots \\ -6 & -3 & 0 & 3 & 6 & \cdots \\ 6 & 0 & -3 & -3 & 0 & \cdots \\ 0 & 6 & 6 & 3 & 0 & \cdots \\ 0 & 0 & 6 & 12 & 15 & \cdots \end{bmatrix}$$

$$(3-6x+6x^2)(1+x) = 3-3x+0x^2+6x^3$$
$$(3-6x+6x^2)(1+x)^2 = 3+0x-3x^2+6x^3+6x^4$$
$$(3-6x+6x^2)(1+x)^3 = 3+3x-3x^2+3x^3+12x^4+6x^5$$
$$(3-6x+6x^2)(1+x)^4 = 3+6x+0x^2+0x^3+15x^4+18x^5+6x^6$$

Clebsch-Gordan Sums

Parameters: $m, n \geq 0$, and $0 \leq k \leq \min(m, n)$

Initial conditions:

For $0 \leq i \leq k$,

$$a_i = (-1)^i \binom{m-i}{k-i} \binom{n-k+i}{i}$$

$$\begin{array}{cccccc} \binom{m}{k} & \binom{m-1}{k-1} & \binom{m-2}{k-2} & \cdots & \binom{m-k+1}{1} & \binom{m-k}{0} \\ \binom{n-k}{0} & \binom{n-k+1}{1} & \binom{n-k+2}{2} & \cdots & \binom{n-1}{k-1} & \binom{n}{k} \end{array}$$

Point: Links $\binom{m}{k}$ to $\binom{n}{k}$

Example

Parameters: $m = 3,\ n = 4,\ k = 2$

$$a_0 = (-1)^0 \binom{3-0}{2-0}\binom{4-2+0}{0} = 3$$
$$a_1 = (-1)^1 \binom{3-1}{2-1}\binom{4-2+1}{1} = -6$$
$$a_2 = (-1)^2 \binom{3-2}{2-2}\binom{4-2+2}{2} = 6$$

Example

$M(m,n,k) \ \to$ truncate to $(m+1)\times(m+n-2k+1)$, and corners

$M(3,4,2) \ \to 4\times 4$

$$\begin{bmatrix} 3 & 3 & 3 & \\ -6 & -3 & 0 & 3 \\ 6 & 0 & -3 & -3 \\ & 6 & 6 & 3 \end{bmatrix}$$

Basic Features of $M(m, n, k)$

True parameters: $m-k$, $n-k$, k

1. $m = (m-k) + k$,
2. $n = (n-k) + k$,
3. $m + n - 2k = (m-k) + (n-k)$

1. Shapes: Usually a hexagon, may degenerate to line segment or parallelogram
2. Opposing side lengths: $n-k+1$, $m-k+1$, $k+1$
3. Constant values on the outer edges:

$$\binom{m}{k}, \quad \binom{n}{k}, \quad \binom{m+n-2k}{m-k}$$

Values on other edges link these values by increasing-decreasing pattern

Example 2

$M(5,4,2)$

Side lengths: 3, 4, 3
Constant values:

$$\binom{5}{2} = 10, \quad \binom{4}{2} = 6, \quad \binom{5+4-2(2)}{5-2} = 10$$

$$\begin{bmatrix} 10 & 10 & 10 & & & \\ -12 & -2 & 8 & 18 & & \\ 6 & -6 & -8 & 0 & 18 & \\ & 6 & 0 & -8 & -8 & 10 \\ & & 6 & 6 & -2 & -10 \\ & & & 6 & 12 & 10 \end{bmatrix}$$

Combinatorial Features of $M(m, n, k)$:

1. Summation formulas,
2. Generating function,
3. Recurrences,
4. Regge Symmetries ($|G| = 72$), and
5. Orthogonality relations.

Generating Function

$$c_{m,n,k}(i,j) = \sum_{l=0}^{k} (-1)^l \binom{i+j-k}{i-l} \binom{m-l}{k-l} \binom{n-k+l}{l}$$

is the $(i+1) \times (i+j-k+1)$-entry of $M(m,n,k)$

Generating Function

For proper coordinate vector entries,

$c_{m,n,k}(i,j)$ is the coefficient of $x^j y^i$ in the expansion of

$$\frac{(x+y)^{i+j-k}}{(1+x)^{m-k+1}(1-y)^{n-k+1}}$$

Regge Symmetry: Example

Weyl Group Symmetry

$$c_{m,n,k}(m-i,n-j) = (-1)^k \, \frac{(m-i)!(n-j)!(i+j-k)!}{i!\, j!\, (m+n-i-j-k)!} \, c_{m,n,k}(i,j)$$

Net effect:

1. rotate $M(m,n,k)$ by 180 degrees,
2. change signs by parity of k, and
3. rescale entries by positive scalar (rational function of five parameters).

Important effect: zero locus is preserved by rotation by 180 degrees

Reverse Recurrence

Reverse Recurrence

$$a_1 \; c_{m,n,k}(i,j) \;=\; a_2 \; c_{m,n,k}(i+1,j) \;+\; a_3 \; c_{m,n,k}(i,j+1)$$

where

1. $a_1 = (i+j-k+1)(m+n-i-j-k)$,
2. $a_2 = (i+1)(m-i)$,
3. $a_3 = (j+1)(n-j)$.

Key Points:

1. upside-down L-pattern, and
2. coefficients are positive when $i < m$, $j < n$.

Hexagonal Symmetries: $D_{12} \cong S_3 \times \mathbb{Z}/2$

Permutations of $m-k$, $n-k$ and k induce transformations between the $M(m,n,k)$.

$$\begin{aligned} \textit{Example}: \quad (m,n,k) = (6,12,4) \quad \rightarrow \quad m-k &= 2, \\ n-k &= 8, \\ k &= 4 \end{aligned}$$

Switch $n-k$ and k:

$$\begin{aligned} m = 2+4 &\rightarrow m = 2+8 = 10, \\ n = 8+4 &\rightarrow n = 4+8 = 12, \\ k = 4 &\rightarrow k = 8 \end{aligned}$$

$M(6,12,4)$ transforms into $M(10,12,8)$, $M(12,10,8)$, $M(12,6,4)$, $M(6,10,2)$, $M(10,6,2)$.

m=6 n=10 k=2

15	15	15	15	15	15	15	15	15	0	0	0	0
-45	-30	-15	0	15	30	45	60	75	90	0	0	0
45	0	-30	-45	-45	-30	0	45	105	180	270	0	0
0	45	45	15	-30	-75	-105	-105	-60	45	225	495	0
0	0	45	90	105	75	0	-105	-210	-270	-225	0	495
0	0	0	45	135	240	315	315	210	0	-270	-495	-495
0	0	0	0	45	180	420	735	1050	1260	1260	990	495

m=12 n=6 k=4

495	495	495	0	0	0	0	0	0	0	0
-495	0	495	990	0	0	0	0	0	0	0
270	-225	-225	270	1260	0	0	0	0	0	0
-90	180	-45	-270	0	1260	0	0	0	0	0
15	-75	105	60	-210	-210	1050	0	0	0	0
0	15	-60	45	105	-105	-315	735	0	0	0
0	0	15	-45	0	105	0	-315	420	0	0
0	0	0	15	-30	-30	75	75	-240	180	0
0	0	0	0	15	-15	-45	30	105	-135	45
0	0	0	0	0	15	0	-45	-15	90	-45
0	0	0	0	0	0	15	15	-30	-45	45
0	0	0	0	0	0	0	15	30	0	-45
0	0	0	0	0	0	0	0	15	45	45

m=6 n=12 k=4

15	15	15	15	15	15	15	15	15	0	0
-90	-75	-60	-45	-30	-15	0	15	30	45	0
270	180	105	45	0	-30	-45	-45	-30	0	45
-495	-225	-45	60	105	105	75	30	-15	-45	-45
495	0	-225	-270	-210	-105	0	75	105	90	45
0	495	495	270	0	-210	-315	-315	-240	-135	-45
0	0	495	990	1260	1260	1050	735	420	180	45

m=6 n=12 k=4

15	15	15	15	15	15	15	15	15	0	0
-90	-75	-60	-45	-30	-15	0	15	30	45	0
270	180	105	45	0	-30	-45	-45	-30	0	45
-495	-225	-45	60	105	105	75	30	-15	-45	-45
495	0	-225	-270	-210	-105	0	75	105	90	45
0	495	495	270	0	-210	-315	-315	-240	-135	-45
0	0	495	990	1260	1260	1050	735	420	180	45

m=12 n=6 k=4

495	495	495	0	0	0	0	0	0	0	0
-495	0	495	990	0	0	0	0	0	0	0
270	-225	-225	270	1260	0	0	0	0	0	0
-90	180	-45	-270	0	1260	0	0	0	0	0
15	-75	105	60	-210	-210	1050	0	0	0	0
0	15	-60	45	105	-105	-315	735	0	0	0
0	0	15	-45	0	105	0	-315	420	0	0
0	0	0	15	-30	-30	75	75	-240	180	0
0	0	0	0	15	-15	-45	30	105	-135	45
0	0	0	0	0	15	0	-45	-15	90	-45
0	0	0	0	0	0	15	15	-30	-45	45
0	0	0	0	0	0	0	15	30	0	-45
0	0	0	0	0	0	0	0	15	45	45

m=6 n=10 k=2

15	15	15	15	15	15	15	15	15	0	0	0	0
-45	-30	-15	0	15	30	45	60	75	90	0	0	0
45	0	-30	-45	-45	-30	0	45	105	180	270	0	0
0	45	45	15	-30	-75	-105	-105	-60	45	225	495	0
0	0	45	90	105	75	0	-105	-210	-270	-225	0	495
0	0	0	45	135	240	315	315	210	0	-270	-495	-495
0	0	0	0	45	180	420	735	1050	1260	1260	990	495

Open Problem

When does $c_{m,n,k}(i,j) = 0$?

Long history:
Wigner, Racah, Regge, Biedenharn, Louck, K. S. Rao, Raynal, et al.

Some techniques include:

1. Special function techniques (traditional)
2. Sums with 2 or 3 terms
3. Regge symmetries
4. Parametrizing special families

Central Value

Suppose m and n are even.
Then $M(m,n,k)$ has odd side lengths $m+1$ by $m+n-2k+1$.

There is a middle entry $c_{m,n,k}(\frac{m}{2},\frac{n}{2})$,
which we call the **central value**.

By Dixon's Identity for binomial summations,

$$c_{m,n,k}(m/2,n/2) = \begin{cases} 0 & k \text{ odd} \\ (-1)^{\frac{k}{2}} \dbinom{\frac{m+n-k}{2}}{\frac{m-k}{2}, \frac{n-k}{2}, \frac{k}{2}} & k \text{ even} \end{cases}$$

Censorship Rule

Censorship Rule

All proper zeros in $M(m, n, k)$ are of the form

$$\begin{pmatrix} \blacksquare & \blacksquare & * \\ \blacksquare & 0 & \blacksquare \\ * & \blacksquare & \blacksquare \end{pmatrix}$$

where $\blacksquare$ must be non-zero.

Proof: If adjacent zeros, use both recurrences:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & 0 & 0 \end{pmatrix} \to \begin{pmatrix} * & * & * \\ * & 0 & * \\ * & 0 & 0 \end{pmatrix} \to \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix},$$

eventually placing a zero on top row in a nonzero position.

Special Cases: Diagonal Belt of Zeros

$M(m, 2k, k)$:

m even, $n = 2k$ and k odd:

$M(m, 2k, k)$ maps to self when switching $n - k$ and k:

1. Fixes absolute values on $\nearrow$ diagonal,
2. Induces sign change $(-1)^k$ on diagonal, and
3. Absolute values fixed under reflection across $\nearrow$ diagonal.

m=6 n=10 k=5

6	6	6	6	6	6	0
-30	-24	-18	-12	-6	0	6
84	54	30	12	0	-6	-6
-168	-84	-30	0	12	12	6
252	84	0	-30	-30	-18	-6
-252	0	84	84	54	24	6
0	-252	-252	-168	-84	-30	-6

Regular Hexagons

$M(2k, 2k, k)$: True parameters $m - k = n - k = k$

Features when $m = n = 2k$ and k odd:

1. Twelve-fold symmetry (regular hexagon with side length $k+1$),
2. Absolute value on three sides is $\binom{2k}{k}$,
3. Three diagonals of zeros (2 broken by censorship), and
4. Distinguished central triangle.

m=10 n=10 k=5

252	252	252	252	252	252	0	0	0	0	0
-756	-504	-252	0	252	504	756	0	0	0	0
1176	420	-84	-336	-336	-84	420	1176	0	0	0
-1176	0	420	336	0	-336	-420	0	1176	0	0
756	-420	-420	0	336	336	0	-420	-420	756	0
-252	504	84	-336	-336	0	336	336	-84	-504	252
0	-252	252	336	0	-336	-336	0	336	252	-252
0	0	-252	0	336	336	0	-336	-336	0	252
0	0	0	-252	-252	84	420	420	84	-252	-252
0	0	0	0	-252	-504	-420	0	420	504	252
0	0	0	0	0	-252	-756	-1176	-1176	-756	-252

252	252	252	252	252	252					
-756	-504	-252	0	252	504	756				
1176	420	-84	-336	-336	-84	420	1176			
-1176	0	420	336	0	-336	-420	0	1176		
756	-420	-420	0	336	336	0	-420	-420	756	
-252	504	84	-336	-336	0	336	336	-84	-504	252
	-252	252	336	0	-336	-336	0	336	252	-252
		-252	0	336	336	0	-336	-336	0	252
			-252	-252	84	420	420	84	-252	-252
				-252	-504	-420	0	420	504	252
					-252	-756	-1176	-1176	-756	-252

Conjectures

When $m = n = 2k$ and

Conjecture 1: k odd: all proper zeros fall on the diagonals.
(True for $k < 2000$.)

Conjecture 2: k even: proper zeros only occur when $k = 8$.
(True for $k < 2000$.)

m=16 n=16 k=8

12870	12870	12870	12870	12870	12870	12870	12870	12870	0	0	0	0	0	0	0	0
-57915	-45045	-32175	-19305	-6435	6435	19305	32175	45045	57915	0	0	0	0	0	0	0
135135	77220	32175	0	-19305	-25740	-19305	0	32175	77220	135135	0	0	0	0	0	0
-212355	-77220	0	32175	32175	12870	-12870	-32175	-32175	0	77220	212355	0	0	0	0	0
245025	32670	-44550	-44550	-12375	19800	32670	19800	-12375	-44550	-44550	32670	245025	0	0	0	0
-212355	32670	65340	20790	-23760	-36135	-16335	16335	36135	23760	-20790	-65340	-32670	212355	0	0	0
135135	-77220	-44550	20790	41580	17820	-18315	-34650	-18315	17820	41580	20790	-44550	-77220	135135	0	0
-57915	77220	0	-44550	-23760	17820	35640	17325	-17325	-35640	-17820	23760	44550	0	-77220	57915	0
12870	-45045	32175	32175	-12375	-36135	-18315	17325	34650	17325	-18315	-36135	-12375	32175	32175	-45045	12870
0	12870	-32175	0	32175	19800	-16335	-34650	-17325	17325	34650	16335	-19800	-32175	0	32175	-12870
0	0	12870	-19305	-19305	12870	32670	16335	-18315	-35640	-18315	16335	32670	12870	-19305	-19305	12870
0	0	0	12870	-6435	-25740	-12870	19800	36135	17820	-17820	-36135	-19800	12870	25740	6435	-12870
0	0	0	0	12870	6435	-19305	-32175	-12375	23760	41580	23760	-12375	-32175	-19305	6435	12870
0	0	0	0	0	12870	19305	0	-32175	-44550	-20790	20790	44550	32175	0	-19305	-12870
0	0	0	0	0	0	12870	32175	32175	0	-44550	-65340	-44550	0	32175	32175	12870
0	0	0	0	0	0	0	12870	45045	77220	77220	32670	-32670	-77220	-77220	-45045	-12870
0	0	0	0	0	0	0	0	12870	57915	135135	212355	245025	212355	135135	57915	12870

Current work:

Use Dixon's Identity with both recursions to find formulas in m, n, k for near central values.

Old work (1993):
Raynal, Rao, et. al. give formula for entries in radius of 4 (more or less).

Techniques includes hypergeometric series, Dixon's Identity, and contiguity relations. Requires working with normalizations.

New work:
Trade hypergeometric series for combinatorial methods:

1. Elementary recursive proof for Dixon's Identity, (4 term recursion)
2. Recursive formulas for central columns, (2 two-term recursions), and
3. Extend to plane using Pascal's Recurrence and D_{12} symmetries.

Thank You!